Published by Institute of Physics Publishing for SISSA/ISAS

Received: *September 2, 2004*
Accepted: *September 16, 2004*

# Two-loop helicity amplitudes for quark-quark scattering in QCD and gluino-gluino scattering in supersymmetric Yang-Mills theory

**Abilio De Freitas***

*Deutsches Elektronen Synchrotron*
*DESY, D-15738 Zeuthen, Germany*
*E-mail:* dfreitas@ifh.de

**Zvi Bern**†

*Department of Physics and Astronomy*
*UCLA, Los Angeles, CA 90095-1547, U.S.A.*
*E-mail:* bern@physics.ucla.edu

Abstract: We present the two-loop QCD helicity amplitudes for quark-quark and quark-antiquark scattering. These amplitudes are relevant for next-to-next-to-leading order corrections to (polarized) jet production at hadron colliders. We give the results in the 't Hooft-Veltman and four-dimensional helicity (FDH) variants of dimensional regularization and present the scheme dependence of the results. We verify that the finite remainder, after subtracting the divergences using Catani's formula, are in agreement with previous results. We also provide the amplitudes for gluino-gluino scattering in pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory. We describe ambiguities in continuing the Dirac algebra to $D$ dimensions, including ones which violate fermion helicity conservation. The finite remainders after subtracting the divergences using Catani's formula, which enter into physical quantities, are free of these ambiguities. We show that in the FDH scheme, for gluino-gluino scattering, the finite remainders satisfy the expected supersymmetry Ward identities.

Keywords: QCD, NLO Computations, Jets, Hadronic Colliders.

*Alexander von Humboldt Fellow.

†Research supported by the US Department of Energy under grant DE-FG03-91ER40662.





JHEP09(2004)039

## Contents



## 1. Introduction

In the coming years, our field is looking forward to the ongoing experiments at the Tevatron at Fermilab and the future ones at the Large Hadron Collider at CERN for unlocking the physics of the electroweak symmetry breaking scale. At hadron colliders, for large momentum transfer, the most copious events are hadronic jets. To explore the validity of the Standard Model at the shortest possible distances, therefore, it would be helpful to determine jet production cross sections with high precision. Calculations of jet production at next-to-leading order (NLO) in the strong coupling constant $\alpha_s$ [1]–[3] agree well with the data over a broad range of transverse momentum. Still, the NLO predictions have an uncertainty from higher-order corrections, which is traditionally estimated using dependence on the renormalization and factorization scales, of order 10% or more. For very large momentum transfer the predictions can be improved by resumming threshold



JHEP09(2004)039

logarithms [4]. There are also sizable uncertainties associated with the experimental input to the parton distribution functions [5]. Nevertheless, an exact next-to-next-to-leading order (NNLO) computation would be desirable. An important step has recently been accomplished with the computation of the three-loop splitting function by Moch, Vermaseren and Vogt [6]. There has also been some earlier work on global fits to the data [7] within an approximate next-to-next-to-leading order (NNLO) framework [8]. Once combined with the matrix elements in complete programs, this should considerably reduce the renormalization and factorization scale uncertainties in production rates. For a summary of the various expected improvements see, for example, ref. [9].

Recent years have seen rapid progress in our ability to compute two-loop matrix elements, especially when there is dependence on more than a single kinematic variable [10]–[23]. Much of this progress has relied on new developments in loop integration [24]–[30] and in understanding the infrared divergences of the theory [31].

An NNLO calculation of jet production requires six-point tree-level, one-loop five-point amplitudes and two-loop four-point amplitudes. The tree amplitudes for six external partons [32, 33] and the one-loop amplitudes for five external partons [34] have been determined some time ago. Anastasiou, Glover, Oleari, and Tejeda-Yeomans have provided the NNLO interferences of the two-loop amplitudes with the tree amplitudes, for all QCD four-parton processes, summed over all external helicities and colors [15]. The helicity amplitudes for $gg \to gg$ were presented in ref. [18]. The $\bar{q}q \to gg$ and $qg \to qg$ helicity amplitudes were presented in refs. [21, 22]. Recently, while preparing this paper, Glover presented the four-quark helicity amplitudes [23]. Here we present the same amplitudes using somewhat different methods, as well as the $\mathcal{N} = 1$ supersymmetric version of these amplitudes. We also describe ambiguities in the amplitudes that first arise in four-fermion amplitudes at NNLO.

For jet production in collisions of unpolarized hadrons, which is the main phenomenological application of the amplitudes, the additional helicity and color information contained in the helicity amplitudes is not necessary. However, for the case of polarized proton scattering at the relativistic heavy ion collider (RHIC) at Brookhaven, the helicity amplitudes are of direct relevance for improving predictions to NNLO accuracy. This may help with the determination of the poorly-known polarized gluon distribution in the proton [35], which is currently available through NLO [36].

Many formal properties of scattering amplitudes are simpler in a helicity basis. As a striking recent example, Witten has linked tree-level helicity amplitudes to a twistor space topological string theory [37] leading to simple [38] and efficient [39] rules for dealing with general [40] tree amplitudes in massless gauge theories. It may also lead to new insights into loop calculations [41]. Other examples are supersymmetry Ward identities [42, 43], the behavior of amplitudes as momenta become collinear [33, 44, 45], and high-energy behavior [46, 47] which all become more evident in a helicity basis. The full color dependence is also helpful for exploring the general structure of infrared singularities [31, 18, 48, 49].

In general, scattering amplitudes with massless gluon exchange possess infrared (soft and collinear) divergences. In dimensional regularization two-loop amplitudes generically contain poles in the dimensional regularization parameter $\epsilon = (4 - D)/2$ up to $1/\epsilon^4$. These singularities have been predicted and organized into a compact form by Catani [31] and cancel from final physical results. As is now standard, we use Catani's formula and color space notation to organize the helicity amplitudes into singular terms (which do contain finite terms in their series expansion in $\epsilon$), plus finite remainders. The precise form of the $1/\epsilon$ poles was not predicted in ref. [31] for general processes at two loops. However, it is now apparent that these terms also have a universal structure depending only on the external legs, based on explicit calculation [12]–[23], matching to resummations [48] and constraints on the functional form as momenta become collinear [49].

A number of dimensional regularization variants are commonly employed for QCD loop calculations. The conventional dimensional regularization (CDR) scheme [50] is usually applied in calculations of amplitude interferences, such as in refs. [51] and [12]–[15]. In the helicity approach, the two commonly used schemes are the 't Hooft-Veltman (HV) scheme [52] and the four-dimensional helicity (FDH) scheme [53, 43]. These schemes differ in the number of polarization components for unobserved gluons. The 't Hooft-Veltman (HV) scheme [52] contains $2-2\epsilon$ virtual gluon states (as does the CDR scheme), whereas the four-dimensional helicity (FDH) scheme [53, 43] assigns 2 states. The FDH scheme is related to dimensional reduction (DR) [54], but is more compatible with the helicity method, because it allows two transverse dimensions in which to define helicity. A more detailed description of the differences between schemes, as well as a definition of the FDH scheme beyond one loop, may be found in ref. [43]. As in ref. [21] we find non-trivial scheme dependence in the finite remainders.

The continuation of the Dirac algebra from four to $D$ dimensions suffers from a variety of ambiguities. One well known ambiguity is in the continuation of $\gamma_5$ [52, 55]. Another lesser known ambiguity, which we also investigate in this paper, is tied to charge conjugation. The appearance of an ambiguity is perhaps not surprising in hindsight given that like $\gamma_5$ in $D=4$ the charge conjugation matrix is a discrete product of $\gamma$ matrices, whose continuation to $D$ dimensions is inherently delicate. In QCD it is a simple matter to sidestep this ambiguity by avoiding use of any charge conjugation identities that are valid only in four dimensions. For $\mathcal{N}=1$ super-Yang-Mills theory, the situation is more complicated, because of the Majorana nature of the gluino implying it is its own antiparticle. At tree level and one loop it turns out to affect the amplitudes only at $\mathcal{O}(\epsilon)$ and is therefore irrelevant through NLO in the coupling constant. Moreover, it is does not appear even at two loops in the previously computed four-gluon or two-fermion two-gluon helicity amplitudes. This ambiguity first becomes relevant at NNLO when there are at least four fermions present. From Catani's infrared divergence formula, an ambiguity in $\mathcal{O}(\epsilon)$ parts of a one-loop amplitude necessarily feeds into an ambiguity in the two-loop amplitude starting at $\mathcal{O}(1/\epsilon)$. Nevertheless, it is reassuring that this ambiguity turns out not to affect the finite remainder of the two-loop amplitudes as long as the different loop orders are computed consistently: We find that the entire ambiguity, including associated finite parts, may be absorbed into the Catani subtraction. Moreover, as noted in ref. [21], when computing a physical process, at NNLO one does not need $\mathcal{O}(\epsilon)$ contributions to the one-loop amplitudes because such terms always cancel. Thus the ambiguity completely cancels from physical quantities, as one may have anticipated from general considerations [56].

For theories without infrared divergences, one can straightforwardly add local counterterms to undo violations of supersymmetry by the regulator. For theories with infrared divergences the situation is more subtle. In any case, it is simpler and more elegant to use a regularization scheme which automatically preserves supersymmetry [54]. The supersymmetry preserving properties of the FDH scheme [53, 43] have been verified explicitly at two loops, for gluon-gluon and gluon-gluino scattering amplitudes [43, 18, 21]. In this paper we study the supersymmetry identities satisfied by gluino-gluino scattering amplitudes. The ambiguities in the gluino-gluino amplitudes, however, causes a difficulty, since the precise value depends on the details of how the calculation was performed. However, after subtracting the singularities using the Catani formula, in the FDH scheme we explicitly show that the finite remainders, which are free of the ambiguities, do satisfy the expected supersymmetry identities.

This paper is arranged as follows. In section 2 we present the helicity and color structure of the four-quark QCD amplitudes. The color space structure of the divergent part of the amplitudes is reviewed in section 3. Section 4 contains the one-loop QCD amplitudes, which appear in Catani's formula for the divergent parts. The one-loop amplitudes are presented in a form valid through $\mathcal{O}(\epsilon^2)$ since these are needed in Catani's formula for the two-loop divergences. The structure of the finite remainders after subtracting out the divergences using Catani's formula is given in section 5. These finite remainders are tabulated in appendix A. The $\mathcal{N}=1$ supersymmetric amplitudes, the ambiguity in their value and the supersymmetry Ward identities, are discussed in section 6. Some auxiliary functions needed for describing the scheme shifts are given in appendix B.

## 2. Helicity and color structure

The three QCD processes considered in this paper are

$$q(p_1,\lambda_1)+\bar{q}(p_2,\lambda_2)\to\bar{Q}(p_3,\lambda_3)+Q(p_4,\lambda_4)\,, \tag{2.1}$$

$$q(p_1,\lambda_1)+\bar{Q}(p_2,\lambda_2)\to q(p_3,\lambda_3)+\bar{Q}(p_4,\lambda_4)\,, \tag{2.2}$$

$$q(p_1,\lambda_1)+Q(p_2,\lambda_2)\to q(p_3,\lambda_3)+Q(p_4,\lambda_4)\,, \tag{2.3}$$

where we use a "standard" convention for the external momentum ($p_i$) and helicity labeling ($\lambda_i$), i.e., particles 1 and 2 are taken to be incoming, while particles 3 and 4 are assumed outgoing. The identical quarks cases are easily obtained from these, as discussed below.

We handle ultraviolet and infrared singularities using dimensional regularization. We consider a continuous set of schemes, labeled by a parameter $\delta_R$ characterizing the number of virtual gluon degrees of freedom circulating in loops. Specifically, when the trace of the Minkowski metric is encountered, we set

$$\eta^\mu{}_\mu\equiv D_s\equiv 4-2\epsilon\,\delta_R\,, \tag{2.4}$$

corresponding to $2(1-\epsilon\,\delta_R)$ gluon states in the loop. Setting $\delta_R=1$ corresponds to the HV scheme [52], while setting $\delta_R=0$ corresponds to the FDH scheme [53, 43].

The CDR and HV schemes have the same standard $\overline{\rm MS}$ coupling constant, $\bar{\alpha}_s(\mu)$. The coupling in a general $\delta_R$ scheme is related to this coupling at NNLO by [43]

$$\alpha_s^{\delta_R}(\mu) = \bar{\alpha}_s(\mu)\Bigg[1 + \frac{C_A}{6}(1-\delta_R)\frac{\bar{\alpha}_s(\mu)}{2\pi} + \left(\frac{C_A^2}{36}(1-\delta_R)^2 + \frac{7C_A^2 - 6C_F T_R N_f}{12}(1-\delta_R)\right) \times \\ \times \left(\frac{\bar{\alpha}_s(\mu)}{2\pi}\right)^2 + \mathcal{O}([\bar{\alpha}_s(\mu)]^3)\Bigg]. \tag{2.5}$$

Henceforth, for simplicity, we suppress the $\delta_R$ index on $\alpha_s(\mu)$.

We work with ultraviolet renormalized amplitudes. The relation between the bare coupling $\alpha_s^u$ and renormalized coupling $\alpha_s(\mu)$, through two-loop order, is [31],

$$\alpha_s^u\, \mu_0^{2\epsilon}\, S_\epsilon = \alpha_s(\mu)\, \mu^{2\epsilon}\left[1 - \frac{\alpha_s(\mu)}{2\pi}\frac{b_0}{\epsilon} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\left(\frac{b_0^2}{\epsilon^2} - \frac{b_1}{2\epsilon}\right) + \mathcal{O}(\alpha_s^3(\mu))\right], \tag{2.6}$$

where $\mu$ is the renormalization scale, $S_\epsilon = \exp[\epsilon(\ln 4\pi + \psi(1))]$, and $\gamma = -\psi(1) = 0.5772\ldots$ is Euler's constant. The first two coefficients appearing in the beta function for QCD, or more generally SU$(N)$ gauge theory with $N_f$ flavors of massless fundamental representation quarks, are scheme-independent,

$$b_0 = \frac{11C_A - 4T_R N_f}{6}, \qquad b_1 = \frac{17C_A^2 - (10C_A + 6C_F)T_R N_f}{6}, \tag{2.7}$$

where $C_A = N$, $C_F = (N^2-1)/(2N)$, and $T_R = 1/2$. (Note that ref. [31] uses the notation $\beta_0 = b_0/(2\pi)$, $\beta_1 = b_1/(2\pi)^2$.)

The perturbative expansion of the $q\bar{q} \to \bar{Q}Q$ amplitude is

$$\mathcal{M}_{q\bar{q}\to\bar{Q}Q}(\alpha_s(\mu), \mu; \{p\}) = 4\pi\alpha_s(\mu)\Bigg[\mathcal{M}^{(0)}_{q\bar{q}\to\bar{Q}Q}(\mu; \{p\}) + \frac{\alpha_s(\mu)}{2\pi}\mathcal{M}^{(1)}_{q\bar{q}\to\bar{Q}Q}(\mu; \{p\}) + \\ + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\mathcal{M}^{(2)}_{q\bar{q}\to\bar{Q}Q}(\mu; \{p\}) + \mathcal{O}(\alpha_s^3(\mu))\Bigg], \tag{2.8}$$

where $\mathcal{M}^{(L)}_{q\bar{q}\to\bar{Q}Q}(\mu; \{p\})$ is the $L^{\rm th}$ loop contribution. The same type of expansion holds, of course, for the $q\bar{Q} \to q\bar{Q}$ and $qQ \to qQ$ amplitudes. Equation (2.6) is equivalent to the following $\overline{\rm MS}$ renormalization prescriptions at one and two loops,

$$\mathcal{M}^{(1)}_{q\bar{q}\to\bar{Q}Q} = S_\epsilon^{-1}\,\mathcal{M}^{(1)\rm unren}_{q\bar{q}\to\bar{Q}Q} - \frac{b_0}{\epsilon}\,\mathcal{M}^{(0)}_{q\bar{q}\to\bar{Q}Q}, \tag{2.9}$$

$$\mathcal{M}^{(2)}_{q\bar{q}\to\bar{Q}Q} = S_\epsilon^{-2}\,\mathcal{M}^{(2)\rm unren}_{q\bar{q}\to\bar{Q}Q} - 2\frac{b_0}{\epsilon}\,S_\epsilon^{-1}\,\mathcal{M}^{(1)\rm unren}_{q\bar{q}\to\bar{Q}Q} + \left(\frac{b_0^2}{\epsilon^2} - \frac{b_1}{2\epsilon}\right)\mathcal{M}^{(0)}_{q\bar{q}\to\bar{Q}Q}. \tag{2.10}$$

We consider the following set of independent helicity configurations $h$

$$h = 1: \quad q(p_1,+) + \bar{q}(p_2,-) \to \bar{Q}(p_3,-) + Q(p_4,+), \tag{2.11}$$

$$h = 2: \quad q(p_1,+) + \bar{q}(p_2,-) \to \bar{Q}(p_3,+) + Q(p_4,-), \tag{2.12}$$

$$h = 3: \quad q(p_1,+) + \bar{Q}(p_2,+) \to q(p_3,+) + \bar{Q}(p_4,+), \tag{2.13}$$

$$h = 4: \quad q(p_1,+) + \bar{Q}(p_2,-) \to q(p_3,+) + \bar{Q}(p_4,-), \tag{2.14}$$

$$h = 5: \quad q(p_1,+) + Q(p_2,+) \to q(p_3,+) + Q(p_4,+), \tag{2.15}$$

$$h = 6: \quad q(p_1,+) + Q(p_2,-) \to q(p_3,+) + Q(p_4,-). \tag{2.16}$$

Other configurations are simply related to these by symmetries. For example, the $q(p_1,-)$ amplitudes are obtained by parity (P), while the $\bar{q}Q \to \bar{q}Q$ and $\bar{q}\bar{Q} \to \bar{q}\bar{Q}$ amplitudes are related to $q\bar{Q} \to q\bar{Q}$ and $qQ \to qQ$, respectively, by an overall charge conjugation (C). (Overall charge conjugation is unaffected by the ambiguity to be discussed in section 6.2.) In defining these helicity configurations we impose helicity conservation on the quark lines.

The cases where both quark lines are identical, i.e. $q\bar{q} \to q\bar{q}$, $qq \to qq$ and $\bar{q}\bar{q} \to \bar{q}\bar{q}$, can also be obtained from configurations (2.11)–(2.16). For example, $q(p_1,+)+\bar{q}(p_2,-) \to \bar{q}(p_3,-) + q(p_4,+)$ can be obtained by taking process (2.11) and subtracting the process (2.14) with $p_3$ and $p_4$ interchanged. The relative minus sign is due to the Fermi statistics. More generally we have for the $qq \to qq$ process,

$$\begin{aligned}
\mathcal{M}_{q_1^+\bar{q}_2^- \to \bar{q}_3^- q_4^+} &= \mathcal{M}_{q_1^+\bar{q}_2^- \to \bar{Q}_3^- Q_4^+} - \mathcal{M}_{q_1^+\bar{Q}_2^- \to q_4^+\bar{Q}_3^-}\,,\\
\mathcal{M}_{q_1^+\bar{q}_2^- \to \bar{q}_3^+ q_4^-} &= \mathcal{M}_{q_1^+\bar{q}_2^- \to \bar{Q}_3^+ Q_4^-}\,,\\
\mathcal{M}_{q_1^+q_2^+ \to q_3^+ q_4^+} &= \mathcal{M}_{q_1^+Q_2^+ \to q_3^+ Q_4^+} - \mathcal{M}_{q_1^+Q_2^+ \to q_4^+ Q_3^+}\,,\\
\mathcal{M}_{q_1^+q_2^- \to q_3^+ q_4^-} &= \mathcal{M}_{q_1^+Q_2^- \to q_3^+ Q_4^-}\,,
\end{aligned} \tag{2.17}$$

where we use $q_1^+$ as a shorthand for $q(p_1,+)$ and so forth.

The color decomposition of the amplitudes is given by

$$\mathcal{M}_h^{(L)} = S_h \times \sum_{c=1}^{2} \mathrm{Tr}^{[c]} \times M_h^{(L),[c]}\,, \qquad h = 1,\dots 6, \tag{2.18}$$

where the elements in the color bases are

$$\mathrm{Tr}^{[1]} = \delta^{i_4}_{\bar{\imath}_1}\delta^{i_2}_{\bar{\imath}_3}\,, \qquad \mathrm{Tr}^{[2]} = \delta^{i_2}_{\bar{\imath}_1}\delta^{i_4}_{\bar{\imath}_3}\,, \qquad h = 1,2, \tag{2.19}$$

$$\mathrm{Tr}^{[1]} = \delta^{i_2}_{\bar{\imath}_1}\delta^{i_3}_{\bar{\imath}_4}\,, \qquad \mathrm{Tr}^{[2]} = \delta^{i_3}_{\bar{\imath}_1}\delta^{i_2}_{\bar{\imath}_4}\,, \qquad h = 3,4, \tag{2.20}$$

$$\mathrm{Tr}^{[1]} = \delta^{i_4}_{\bar{\imath}_1}\delta^{i_3}_{\bar{\imath}_2}\,, \qquad \mathrm{Tr}^{[2]} = \delta^{i_3}_{\bar{\imath}_1}\delta^{i_4}_{\bar{\imath}_2}\,, \qquad h = 5,6. \tag{2.21}$$

In our previous calculations of the $gg \to gg$ and $q\bar{q} \to gg$ processes [18, 21], traces of products of color matrices appeared. For consistency of notation, we have maintained here the "Tr" notation for the color bases, even though there are no traces in the present case.

The helicity-dependent, phase-containing factors $S_h$ arise from evaluating the amplitudes in the spinor helicity formalism [57]. They are,

$$\begin{aligned}
S_1 &= -i\frac{\langle 3\,1\rangle}{\langle 4\,2\rangle}\,, \qquad S_2 = -i\frac{\langle 4\,1\rangle}{\langle 3\,2\rangle}\,, \qquad S_3 = -i\frac{\langle 2\,1\rangle}{\langle 4\,3\rangle}\,,\\
S_4 &= -i\frac{\langle 4\,1\rangle}{\langle 3\,2\rangle}\,, \qquad S_5 = -i\frac{\langle 2\,1\rangle}{\langle 4\,3\rangle}\,, \qquad S_6 = -i\frac{\langle 4\,1\rangle}{\langle 2\,3\rangle}\,.
\end{aligned} \tag{2.22}$$

The spinor inner products [57, 33] are $\langle i\,j\rangle = \langle i^-|j^+\rangle$ and $[i\,j] = \langle i^+|j^-\rangle$, where $|i^\pm\rangle$ are massless Weyl spinors of momentum $k_i$, labeled with the sign of the helicity. They are anti-symmetric, with norm $|\langle i\,j\rangle| = |[i\,j]| = \sqrt{s_{ij}}$, where $s_{ij} = 2k_i\cdot k_j$. Notice that the prefactors $S_h$ are all pure phases, i.e.,

$$|S_h|^2 = 1\,, \qquad h = 1\dots 6\,. \tag{2.23}$$

The quantities $M_h^{(0),[c]}$ depend only on the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_4)^2$ and $u = (p_1 - p_3)^2$. At tree level in the color bases (2.19), (2.20) and (2.21) they are given by,

$$\begin{aligned}
M_1^{(0),[1]} &= \frac{u}{s}\,, \qquad M_2^{(0),[1]} = \frac{t}{s}\,, \qquad M_3^{(0),[1]} = \frac{s}{u}\,,\\
M_4^{(0),[1]} &= \frac{t}{u}\,, \qquad M_5^{(0),[1]} = \frac{s}{u}\,, \qquad M_6^{(0),[1]} = \frac{t}{u}\,,\\
M_h^{(0),[2]} &= -\frac{1}{N} M_h^{(0),[1]}\,, \qquad h = 1\ldots 6\,.
\end{aligned} \tag{2.24}$$

## 3. Infrared singularities

We now briefly review the structure of the soft and collinear singularities of dimensionally regularized one- and two-loop QCD amplitudes, using Catani's color space notation [31]. The finite remainders are given in section 4.2, section 5.2 and appendix A.

The infrared divergences of renormalized one- and two-loop $n$-point amplitudes are given by [58, 31],

$$|\mathcal{M}_n^{(1)}(\mu;\{p\})\rangle_{\rm RS} = \boldsymbol{I}^{(1)}(\epsilon,\mu;\{p\})\; |\mathcal{M}_n^{(0)}(\mu;\{p\})\rangle_{\rm RS} + |\mathcal{M}_n^{(1)\rm fin}(\mu;\{p\})\rangle_{\rm RS}\,, \tag{3.1}$$

$$\begin{aligned}
|\mathcal{M}_n^{(2)}(\mu;\{p\})\rangle_{\rm RS} &= \boldsymbol{I}^{(1)}(\epsilon,\mu;\{p\})\; |\mathcal{M}_n^{(1)}(\mu;\{p\})\rangle_{\rm RS} + \\
&\quad + \boldsymbol{I}^{(2)}_{\rm RS}(\epsilon,\mu;\{p\})\; |\mathcal{M}_n^{(0)}(\mu;\{p\})\rangle_{\rm RS} + |\mathcal{M}_n^{(2)\rm fin}(\mu;\{p\})\rangle_{\rm RS}\,,
\end{aligned} \tag{3.2}$$

where the "ket" notation $|\mathcal{M}_n^{(L)}(\mu;\{p\})\rangle_{\rm RS}$ indicates that the $L$-loop amplitude is treated as a vector in color space. The components of this vector are given by the $M_h^{(L),[c]}$ in eq. (2.18). The subscript RS indicates that a quantity depends on the choice of regularization and renormalization scheme. The divergences of $\mathcal{M}_n^{(1)}$ are encoded in the color operator $\boldsymbol{I}^{(1)}$, while those of $\mathcal{M}_n^{(2)}$ also involve the scheme-dependent operator $\boldsymbol{I}^{(2)}_{\rm RS}$.

The operator $\boldsymbol{I}^{(1)}$, which controls the one-loop singularity structure, is given by

$$\boldsymbol{I}^{(1)}(\epsilon,\mu;\{p\}) = \frac{1}{2}\frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \boldsymbol{T}_i\cdot\boldsymbol{T}_j \left[\frac{1}{\epsilon^2} + \frac{\gamma_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right] \left(\frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2p_i\cdot p_j}\right)^{\epsilon}, \tag{3.3}$$

where $\lambda_{ij} = +1$ if $i$ and $j$ are both incoming or outgoing partons, and $\lambda_{ij} = 0$ otherwise. The color charge $\boldsymbol{T}_i = \{T_i^a\}$ is a vector with respect to the generator label $a$, and an $SU(N)$ matrix with respect to the color indices of the outgoing parton $i$. For external fermions, the ratio

$$\frac{\gamma_q}{\boldsymbol{T}_i^2} = \frac{3}{2}\,, \tag{3.4}$$

is independent of the representation. For quarks, $\boldsymbol{T}_i^2 = C_F = (N^2-1)/(2N)$; for gluinos $\boldsymbol{T}_i^2 = C_A = N$. The two-loop operator $\boldsymbol{I}^{(2)}_{\rm RS}$ is [31]

$$\begin{aligned}
\boldsymbol{I}^{(2)}_{\rm RS}(\epsilon,\mu;\{p\}) &= -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon,\mu;\{p\})\left(\boldsymbol{I}^{(1)}(\epsilon,\mu;\{p\}) + \frac{2b_0}{\epsilon}\right) + \\
&\quad + \frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{b_0}{\epsilon} + K_{\rm RS}\right)\boldsymbol{I}^{(1)}(2\epsilon,\mu;\{p\}) + \\
&\quad + \boldsymbol{H}^{(2)}_{\rm RS}(\epsilon,\mu;\{p\})\,,
\end{aligned} \tag{3.5}$$



JHEP09(2004)039

where the coefficient $K_{\rm RS}$ depends on $\delta_R$ and is given by [31, 18]

$$K_{\rm RS} = \left[\frac{67}{18} - \frac{\pi^2}{6} - \left(\frac{1}{6} + \frac{4}{9}\epsilon\right)(1-\delta_R)\right] C_A - \frac{10}{9} T_R N_f \,. \tag{3.6}$$

The function $\boldsymbol{H}^{(2)}_{\rm RS}$ contains only *single* poles, and splits into two types of terms,

$$\boldsymbol{H}^{(2)}(\epsilon) = \frac{e^{-\epsilon\psi(1)}}{4\epsilon\,\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^{2\epsilon}\left(4H_q^{(2)}\,\mathbf{1} + \hat{\boldsymbol{H}}^{(2)}\right), \tag{3.7}$$

where a standard analytic continuation is needed to bring $s$ to the physical region. We find that the term proportional to the identity matrix in color space $\mathbf{1}$ is given by

$$\begin{aligned} H_q^{(2)} = {} & \left(\frac{13}{2}\zeta_3 - \frac{23}{48}\pi^2 + \frac{245}{216}\right) C_A C_F + \left(-6\zeta_3 + \frac{\pi^2}{2} - \frac{3}{8}\right) C_F^2 + \left(\frac{\pi^2}{12} - \frac{25}{54}\right) C_F T_R N_f + \\ & + \left(-\frac{4}{3} C_A C_F + \frac{1}{2} C_F^2 + \frac{1}{6} C_F T_R N_f\right)(1-\delta_R) \,. \end{aligned} \tag{3.8}$$

This term survives the sum over colors, and the expression for $H_q^{(2)}$ in the HV scheme ($\delta_R = 1$) agrees, as expected, with previous color-summed results in the CDR scheme [13]–[15] and [59].

The second term in $\boldsymbol{H}^{(2)}(\epsilon)$ has exactly the same type of nontrivial color and kinematic dependence found in the $gg \to gg$ and $q\bar{q} \to gg$ helicity amplitudes [18, 21], namely

$$\hat{\boldsymbol{H}}^{(2)} = -4 \ln\left(\frac{-s}{-t}\right) \ln\left(\frac{-t}{-u}\right) \ln\left(\frac{-u}{-s}\right) \times \left[\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 \,, \boldsymbol{T}_2 \cdot \boldsymbol{T}_3\right], \tag{3.9}$$

where again appropriate analytic continuation are required. For example, in the $s$-channel, $\ln((-s)/(-t)) \to \ln s - \ln(-t) - i\pi$. In refs. [18, 21, 22] it was shown that for the $gg \to gg$ and $q\bar{q} \to gg$ amplitudes the structure of this term is independent of the helicity configuration, and whether the external legs are quarks or gluons. Here once again, we find this to be the case. An ansatz generalizing $\hat{\boldsymbol{H}}^{(2)}$ for an arbitrary number of external legs has been presented recently in [49]. Because of the commutator structure, $\hat{\boldsymbol{H}}^{(2)}$ vanishes when sandwiched between tree amplitudes, after performing a sum over colors; hence it drops out of the color-summed interference of the two-loop amplitudes with the tree amplitudes [18, 21].

For each color basis we will have a different $\boldsymbol{I}^{(1)}$ matrix. For the basis (2.19) we have,

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)}\,\xi_q \begin{pmatrix} 2C_F\mathtt{T} - \frac{1}{N}(\mathtt{S}-\mathtt{U}) & \mathtt{T}-\mathtt{U} \\ \mathtt{S}-\mathtt{U} & 2C_F\mathtt{S} - \frac{1}{N}(\mathtt{T}-\mathtt{U}) \end{pmatrix}, \tag{3.10}$$

where

$$\mathtt{S} = \left(\frac{\mu^2}{-s}\right)^{\epsilon}, \qquad \mathtt{T} = \left(\frac{\mu^2}{-t}\right)^{\epsilon}, \qquad \mathtt{U} = \left(\frac{\mu^2}{-u}\right)^{\epsilon}, \qquad \xi_q = \frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\,. \tag{3.11}$$

The corresponding operator for $q\bar{Q} \to q\bar{Q}$ in the basis (2.20) is obtained by changing $\mathtt{S} \to \mathtt{U}$, $\mathtt{T} \to \mathtt{S}$ and $\mathtt{U} \to \mathtt{T}$ in eq. (3.10). Similarly, the operator for $qQ \to qQ$ in the basis (2.21) is obtained by exchanging $\mathtt{S}$ and $\mathtt{U}$ in (3.10).

A typical partonic cross section requires an amplitude interference, summed over all external colors. Such interferences are evaluated in the color bases (2.19)–(2.21) as

$$\mathcal{I}^{(L,L')}_{\lambda_1\lambda_2\lambda_3\lambda_4} \equiv \langle \mathcal{M}^{(L)}_{\lambda_1\lambda_2\lambda_3\lambda_4} | \mathcal{M}^{(L')}_{\lambda_1\lambda_2\lambda_3\lambda_4} \rangle = \sum_{c,c'=1}^{2} M^{(L),[c]*}_{\lambda_1\lambda_2\lambda_3\lambda_4} \mathcal{C}_{cc'} M^{(L'),[c']}_{\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{3.12}$$

where the symmetric matrix $\mathcal{C}_{cc'} \equiv \sum_{\text{colors}} \text{Tr}^{[c]*}\text{Tr}^{[c']}$ is

$$\mathcal{C} = \begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix}. \tag{3.13}$$

## 4. One-loop QCD amplitudes

The one-loop amplitudes for $q\bar{q} \to \bar{Q}Q$ were first evaluated through $\mathcal{O}(\epsilon^0)$ as an interference with the tree amplitude in the CDR scheme [51]. Later they were evaluated as helicity amplitudes in the HV, FDH and $\overline{\text{DR}}$ schemes [60].

Because $\boldsymbol{I}^{(1)}$ contains terms of order $1/\epsilon^2$, the $\boldsymbol{I}^{(1)}|\mathcal{M}^{(1)}\rangle_{\text{R.S.}}$ term in the infrared decomposition (3.2) of the two-loop $q\bar{q} \to \bar{Q}Q$ amplitudes requires the series expansion of the one-loop amplitudes through $\mathcal{O}(\epsilon^2)$. In section 4.1 we present the all-order results in the color bases (2.19)–(2.21), with the normalizations implicit in eq. (2.8), in terms of integral functions whose series expansions have been evaluated to the required order [11, 12].

In ref. [18] it was shown that the $\mathcal{O}(\epsilon)$ terms in one-loop amplitudes such as $\mathcal{M}^{(1)}_{q\bar{q}\to\bar{Q}Q}$ are not required for the construction of a numerical NNLO program, once the divergences have been subtracted from $\mathcal{M}^{(2)}_{q\bar{q}\to\bar{Q}Q}$ using Catani's formula. Thus we need only present explicit formulæ for the $\mathcal{O}(\epsilon^0)$ finite remainders $\mathcal{M}^{(1)\text{fin}}_{q\bar{q}\to\bar{Q}Q}$ of the one-loop amplitudes, after ultraviolet renormalization (2.9) and subtraction of infrared divergences (3.1). We do this in section 4.2, for a general $\delta_R$ scheme. The formula (5.10) for converting the two-loop finite remainders $\mathcal{M}^{(2)\text{fin}}_{q\bar{q}\to\bar{Q}Q}$ from one scheme to another are most compactly presented in terms of the $\delta_R$-dependent parts of the one-loop amplitudes at order $\epsilon$; the explicit values of these quantities are collected in appendix B.

### 4.1 One-loop all orders in $\epsilon$ QCD amplitudes

We now present the one-loop $q\bar{q} \to \bar{Q}Q$ amplitudes in the color bases (2.19)–(2.21), with the normalizations implicit in eq. (2.8), in a form valid to all orders in $\epsilon$.

At one loop the crossing properties of the amplitudes are relatively simple, so we present the explicit values of the helicity amplitudes for the process $q\bar{q} \to \bar{Q}Q$. The process $q\bar{Q} \to q\bar{Q}$ may be obtained from these by crossing the initial antiquark and final quark into the final and initial states respectively, and the process $qQ \to qQ$ may be obtained by crossing antiquark 2 into the final state, and antiquark 3 into the initial state,

$$M_3^{(1),[c]}(s,t,u) = M_2^{(1),[c]}(u,s,t), \qquad c = 1,2, \tag{4.1}$$
$$M_4^{(1),[c]}(s,t,u) = M_1^{(1),[c]}(u,s,t), \qquad c = 1,2, \tag{4.2}$$
$$M_5^{(1),[c]}(s,t,u) = M_1^{(1),[c]}(u,t,s), \qquad c = 1,2, \tag{4.3}$$
$$M_6^{(1),[c]}(s,t,u) = M_2^{(1),[c]}(u,t,s), \qquad c = 1,2, \tag{4.4}$$

where $M_h^{(L),[c]}$ is defined in eq. (2.18) with the color bases (2.19), (2.20) and (2.21) using the helicity configurations $h$ defined in eqs. (2.11)-(2.16). After crossing, appropriate analytic continuations are required to bring each function into the physical region.

A compact representation for the unrenormalized amplitudes is,

$$M_1^{(1),[1]}(s,t,u) = A_+(s,u,t) + \frac{2}{N}A_1(s,t,u) + \left(N - \frac{2}{N}\right)A_2(s,t,u)\,, \tag{4.5}$$

$$M_1^{(1),[2]}(s,t,u) = -\frac{1}{N}A_+(s,u,t) - \left(1 + \frac{1}{N^2}\right)A_1(s,t,u) + \frac{1}{N^2}A_2(s,t,u)\,, \tag{4.6}$$

$$M_2^{(1),[1]}(s,t,u) = -A_-(s,t,u) + \frac{2}{N}A_3(s,t,u) + \left(N - \frac{2}{N}\right)A_4(s,t,u)\,, \tag{4.7}$$

$$M_2^{(1),[2]}(s,t,u) = \frac{1}{N}A_-(s,t,u) - \left(1 + \frac{1}{N^2}\right)A_3(s,t,u) + \frac{1}{N^2}A_4(s,t,u)\,, \tag{4.8}$$

where,

$$A_\pm(s,t,u) = \frac{1}{2\epsilon-1}\left[\frac{N}{2}\epsilon^3\delta_R^2 s \pm \left(\frac{\epsilon(1-\epsilon\delta_R)}{2(\epsilon-1)}\left(\frac{\epsilon-2}{2\epsilon-3}N + \frac{2\epsilon-1}{N}\right) + \right.\right.$$
$$\left.\left. + \frac{\epsilon(\epsilon-1)}{2\epsilon-3}N_f - \frac{1}{N}\right)t\right]\mathrm{Tri}^{(4)}(s)\,, \tag{4.9}$$

$$A_1(s,t,u) = \frac{\epsilon^2}{2(2\epsilon-1)}\delta_R s\,\mathrm{Tri}^{(4)}(s) - \left[\frac{\epsilon^2}{2(2\epsilon-1)}\delta_R(1+\epsilon\delta_R)u + \frac{u^2}{s}\right]\mathrm{Tri}^{(4)}(u) -$$
$$- \left[\frac{\epsilon^2}{2}\delta_R(1+\epsilon\delta_R)s - \left(\epsilon^2\delta_R - (2\epsilon-1)\frac{t}{s}\right)u\right]\mathrm{Box}^{(6)}(s,u)\,, \tag{4.10}$$

$$A_2(s,t,u) = \frac{\epsilon}{2(2\epsilon-1)}s\,\mathrm{Tri}^{(4)}(s) - \left[\frac{\epsilon}{2\epsilon-1}(1+\epsilon^2\delta_R^2) + 2\frac{u}{s}\right]\frac{t}{2}\mathrm{Tri}^{(4)}(t) +$$
$$+ \left[\frac{1-\epsilon}{2}(1+\epsilon^2\delta_R^2)s - \left(\epsilon^2\delta_R + (1-2\epsilon)\frac{t}{s}\right)u\right]\mathrm{Box}^{(6)}(s,t)\,, \tag{4.11}$$

$$A_3(s,t,u) = \left[\frac{\epsilon}{2}(\epsilon-1)\delta_R(1+\epsilon\delta_R)s + \epsilon^2\delta_R t - \epsilon\frac{t^2}{s} + (1-\epsilon)\frac{(u^2+t^2)}{2s}\right]\mathrm{Box}^{(6)}(s,u) +$$
$$+ \left[\frac{\epsilon^2}{2}\delta_R(1+\epsilon\delta_R) + (1-2\epsilon)\frac{t}{s} - \frac{\epsilon}{2}\right]\frac{u}{2\epsilon-1}\mathrm{Tri}^{(4)}(u) +$$
$$+ \frac{\epsilon(1-\epsilon\delta_R)}{2(2\epsilon-1)}s\,\mathrm{Tri}^{(4)}(s)\,, \tag{4.12}$$

$$A_4(s,t,u) = \left[\frac{\epsilon^3}{2(2\epsilon-1)}\delta_R^2 t - \frac{t^2}{s}\right]\mathrm{Tri}^{(4)}(t) +$$
$$+ \left[\frac{\epsilon^3}{2}\delta_R^2 s - \epsilon^2\delta_R t + ((1-\epsilon)u + \epsilon t)\frac{t}{s}\right]\mathrm{Box}^{(6)}(s,t)\,. \tag{4.13}$$

Here $\mathrm{Tri}^{(4)}(s)$ is the scalar triangle integral in $4-2\epsilon$ dimensions with one external massive leg, and $\mathrm{Box}^{(6)}(s,t)$ is the all-massless scalar box integral in $6-2\epsilon$ dimensions. The expansion of these integrals to $\mathcal{O}(\epsilon^2)$ in the various kinematic channels is given, for example, in refs. [12, 16]. The renormalized amplitudes are obtained by subtracting $b_0 M_h^{(0),[c]}/\epsilon$ from each of eqs. (4.5)–(4.8), where $c, h$ correspond to the amplitude under consideration.

### 4.2 Finite remainders

We now give the finite remainders of the one-loop $q\bar{q} \to \bar{Q}Q$, $q\bar{Q} \to q\bar{Q}$ and $qQ \to qQ$ amplitudes at $\mathcal{O}(\epsilon^0)$, defined by $\mathcal{M}^{(1)\mathrm{fin}}_{q\bar{q}\to gg}$ and $\mathcal{M}^{(1)\mathrm{fin}}_{qg\to gq}$ in eq. (3.1) and color decomposed into $M_h^{(1),[c]\mathrm{fin}}$ in eq. (2.18). We write,

$$\begin{aligned} M_h^{(1),[1]\mathrm{fin}} &= \left[-b_0\left(\ln\left(\frac{s}{\mu^2}\right) - i\pi\right) + \left(\frac{N}{3} - \frac{1}{2N}\right)(1-\delta_R)\right] M_h^{(0),[1]} + \\ &\quad + N a_h^{[1]} + \frac{1}{N} b_h^{[1]} + N_f d_h^{[1]} , \end{aligned} \tag{4.14}$$

$$\begin{aligned} M_h^{(1),[2]\mathrm{fin}} &= \left[-b_0\left(\ln\left(\frac{s}{\mu^2}\right) - i\pi\right) + \left(\frac{N}{3} - \frac{1}{2N}\right)(1-\delta_R)\right] M_h^{(0),[2]} + \\ &\quad + h_h^{[2]} + \frac{N_f}{N} j_h^{[2]} + \frac{1}{N^2} k_h^{[2]} . \end{aligned} \tag{4.15}$$

For the $h = 1$ helicity amplitude, the independent remainder functions $a$, $b$, $d$, $h$, $j$ and $k$ are

$$a_1^{[1]} = \frac{13}{18}y + \left(\frac{x}{2} - y\right)X + \frac{1}{4}\left(y + \frac{x^2}{y}\right)X^2 + i\pi\left[\frac{x}{2} - y + \frac{1}{2}\left(\frac{x^2}{y} + y\right)X\right], \tag{4.16}$$

$$\begin{aligned} b_1^{[1]} &= (2y - x)X - \frac{1}{2}\left(\frac{x^2}{y} + y\right)X^2 + y(Y^2 - 3Y + 4) + \\ &\quad + i\pi\left[1 - \left(\frac{x^2}{y} + y\right)X + 2yY\right], \end{aligned} \tag{4.17}$$

$$d_1^{[1]} = -\frac{5}{9}y , \tag{4.18}$$

$$h_1^{[2]} = -\frac{y}{2}\left(Y^2 - 3Y + \frac{13}{9}\right) - i\pi y\left(Y - \frac{3}{2}\right), \tag{4.19}$$

$$j_1^{[2]} = \frac{5}{9}y , \tag{4.20}$$

$$\begin{aligned} k_1^{[2]} &= \left(\frac{x}{2} - y\right)X + \frac{1}{4}\left(\frac{x^2}{y} + y\right)X^2 + y\left(\frac{3}{2}Y - \frac{1}{2}Y^2 - 4\right) - \\ &\quad - i\pi\left[\frac{1}{2} - \frac{1}{2}\left(\frac{x^2}{y} + y\right)X + yY\right], \end{aligned} \tag{4.21}$$

where

$$x = \frac{t}{s}, \qquad y = \frac{u}{s}, \qquad X = \ln(-x), \qquad Y = \ln(-y) . \tag{4.22}$$

For $h = 2$ the functions are

$$a_2^{[1]} = \frac{x}{2}\left(X^2 - 3X + \frac{13}{9}\right) + i\pi x\left(X - \frac{3}{2}\right), \tag{4.23}$$

$$\begin{aligned} b_2^{[1]} &= x(4 + 3X - X^2) + \frac{1}{2}\left(\frac{y^2}{x} + x\right)Y^2 + (y - 2x)Y - \\ &\quad - i\pi\left[1 + 2xX - \left(x + \frac{y^2}{x}\right)Y\right], \end{aligned} \tag{4.24}$$

$$d_2^{[1]} = -\frac{5}{9}x , \tag{4.25}$$

$$h_2^{[2]} = -\frac{13}{18}x - \frac{1}{4}\left(\frac{y^2}{x} + x\right)Y^2 - \left(\frac{y}{2} - x\right)Y - i\pi\left[\frac{y}{2} - x + \frac{1}{2}\left(x + \frac{y^2}{x}\right)Y\right], \quad (4.26)$$

$$j_2^{[2]} = \frac{5}{9}x\,, \quad (4.27)$$

$$k_2^{[2]} = -\frac{1}{4}\left(\frac{y^2}{x} + x\right)Y^2 - \left(\frac{y}{2} - x\right)Y + x\left(\frac{1}{2}X^2 - 4 - \frac{3}{2}X\right) + \\ + i\pi\left[\frac{1}{2} + xX - \frac{1}{2}\left(x + \frac{y^2}{x}\right)Y\right]. \quad (4.28)$$

For $h = 3$ the functions are

$$a_3^{[1]} = \left(\frac{1}{2}Y^2 - \frac{1}{3}Y + \frac{13}{18}\right)\frac{1}{y} + i\pi\left(Y - \frac{1}{3}\right)\frac{1}{y}\,, \quad (4.29)$$

$$b_3^{[1]} = \left[(1+x^2)\left(\frac{\pi^2}{2} - XY + \frac{1}{2}X^2\right) + 4 - (2-x)X + \frac{1}{2}(x^2-1)Y^2\right]\frac{1}{y} + Y - \\ - i\pi(2Y+3)\frac{1}{y}\,, \quad (4.30)$$

$$d_3^{[1]} = \left(\frac{1}{3}Y - \frac{5}{9}\right)\frac{1}{y} + i\frac{\pi}{3y}\,, \quad (4.31)$$

$$h_3^{[2]} = \left[(1+x^2)\left(\frac{1}{2}XY - \frac{1}{4}(\pi^2 + Y^2 + X^2)\right) + \left(1 - \frac{x}{2}\right)X + \frac{1}{3}Y - \frac{13}{18}\right]\frac{1}{y} - \\ - \frac{1}{2}Y + i\pi\frac{11}{6y}\,, \quad (4.32)$$

$$j_3^{[2]} = \left(\frac{5}{9} - \frac{1}{3}Y\right)\frac{1}{y} - i\frac{\pi}{3y}\,, \quad (4.33)$$

$$k_3^{[2]} = \left[\frac{1}{4}(1+x^2)(2XY - \pi^2 - X^2) - 4 + \left(1 - \frac{x}{2}\right)X + \frac{1}{4}(1-x^2)Y^2\right]\frac{1}{y} - \frac{1}{2}Y + \\ + i\pi\left(Y + \frac{3}{2}\right)\frac{1}{y}. \quad (4.34)$$

For $h = 4$ the functions are

$$a_4^{[1]} = \left[x\left(\frac{13}{18} - \frac{1}{3}Y\right) + \frac{1}{4}\left(x + \frac{1}{x}\right)Y^2\right]\frac{1}{y} + \frac{1}{2}Y - \\ - i\pi\left[\frac{5}{3}x + 1 - \left(\frac{1}{x} + x\right)Y\right]\frac{1}{2y}\,, \quad (4.35)$$

$$b_4^{[1]} = \left[x(X^2 + \pi^2 + 4 - 3X - 2XY) - \frac{1}{2}\left(\frac{1}{x} - x\right)Y^2\right]\frac{1}{y} - Y - \\ - i\pi\left[2x - 1 + \left(x + \frac{1}{x}\right)Y\right]\frac{1}{y}\,, \quad (4.36)$$

$$d_4^{[1]} = \left(\frac{1}{3}Y - \frac{5}{9}\right)\frac{x}{y} + i\pi\frac{x}{3y}\,, \quad (4.37)$$

$$h_4^{[2]} = \left[-\frac{\pi^2}{2} + \frac{3}{2}X + \frac{1}{3}Y - \frac{1}{2}(X^2 + Y^2) + XY - \frac{13}{18}\right]\frac{x}{y} + i\pi\frac{11x}{6y}\,, \quad (4.38)$$

$$j_4^{[2]} = -\left(\frac{1}{3}Y - \frac{5}{9}\right)\frac{x}{y} - i\pi\frac{x}{3y}\,, \quad (4.39)$$

$$k_4^{[2]} = -\left[x\left(\frac{\pi^2}{2}+4-\frac{3}{2}X+\frac{1}{2}X^2-XY\right)-\frac{1}{4}\left(\frac{1}{x}-x\right)Y^2\right]\frac{1}{y}+\frac{1}{2}Y+$$
$$+i\pi\left[x-\frac{1}{2}+\frac{1}{2}\left(\frac{1}{x}+x\right)Y\right]\frac{1}{y}\,. \tag{4.40}$$

For $h=5$ the functions are

$$a_5^{[1]} = \left[\frac{1}{4}(1+x^2)(\pi^2-2XY+Y^2+X^2)+\frac{13}{18}-\left(1-\frac{x}{2}\right)X-\frac{1}{3}Y\right]\frac{1}{y}+$$
$$+\frac{1}{2}Y-i\pi\frac{11}{6y}\,, \tag{4.41}$$

$$b_5^{[1]} = \left[\frac{1}{2}(1+x^2)(2XY-X^2-\pi^2)+4-(x-2)X+\frac{1}{2}(1-x^2)Y^2\right]\frac{1}{y}-$$
$$-Y+i\pi(2Y+3)\frac{1}{y}\,, \tag{4.42}$$

$$d_5^{[1]} = \left(\frac{1}{3}Y-\frac{5}{9}\right)\frac{1}{y}+i\frac{\pi}{3y}\,, \tag{4.43}$$

$$h_5^{[2]} = -\left(\frac{1}{2}Y^2-\frac{1}{3}Y+\frac{13}{18}\right)\frac{1}{y}-i\pi\left(Y-\frac{1}{3}\right)\frac{1}{y}\,, \tag{4.44}$$

$$j_5^{[2]} = -\left(\frac{1}{3}Y-\frac{5}{9}\right)\frac{1}{y}-i\frac{\pi}{3y}\,, \tag{4.45}$$

$$k_5^{[2]} = \left[\frac{1}{4}(1+x^2)(X^2+\pi^2-2XY)-4+\left(\frac{x}{2}-1\right)X-\frac{1}{4}(1-x^2)Y^2\right]\frac{1}{y}+$$
$$+\frac{1}{2}Y-i\pi\left(Y+\frac{3}{2}\right)\frac{1}{y}\,. \tag{4.46}$$

For $h=6$ the functions are

$$a_6^{[1]} = \left[\frac{\pi^2}{2}-\frac{3}{2}X-\frac{1}{3}Y+\frac{1}{2}(X^2+Y^2)-XY+\frac{13}{18}\right]\frac{x}{y}-i\pi\frac{11x}{6y}\,, \tag{4.47}$$

$$b_6^{[1]} = \left[x(4-\pi^2+3X-X^2+2XY)+\frac{1}{2}\left(\frac{1}{x}-x\right)Y^2\right]\frac{1}{y}+Y+$$
$$+i\pi\left[2x-1+\left(x+\frac{1}{x}\right)Y\right]\frac{1}{y}\,, \tag{4.48}$$

$$d_6^{[1]} = \left(\frac{1}{3}Y-\frac{5}{9}\right)\frac{x}{y}+i\pi\frac{x}{3y}\,, \tag{4.49}$$

$$h_6^{[2]} = \left[x\left(\frac{1}{3}Y-\frac{13}{18}\right)-\frac{1}{4}\left(x+\frac{1}{x}\right)Y^2\right]\frac{1}{y}-\frac{1}{2}Y+$$
$$+i\pi\left[\frac{5}{6}x+\frac{1}{2}-\frac{1}{2}\left(\frac{1}{x}+x\right)Y\right]\frac{1}{y}\,, \tag{4.50}$$

$$j_6^{[2]} = \left(\frac{5}{9}-\frac{1}{3}Y\right)\frac{x}{y}-i\pi\frac{x}{3y}\,, \tag{4.51}$$

$$k_6^{[2]} = \left[x\left(\frac{\pi^2}{2}-4-\frac{3}{2}X+\frac{1}{2}X^2-XY\right)-\frac{1}{4}\left(\frac{1}{x}-x\right)Y^2\right]\frac{1}{y}-\frac{1}{2}Y-$$
$$-i\pi\left[x-\frac{1}{2}+\frac{1}{2}\left(\frac{1}{x}+x\right)Y\right]\frac{1}{y}\,. \tag{4.52}$$

For the HV scheme ($\delta_R = 1$), the results (4.14)–(4.52) for the finite remainders of the one-loop helicity amplitudes are in agreement with those of ref. [23].

## 5. Two-loop QCD amplitudes and finite remainders

### 5.1 Construction of amplitudes

We generated the Feynman graphs for $q\bar{q} \to \bar{Q}Q$ using `QGRAF` [61], from which a `MAPLE` program was constructed to evaluate each graph. We employed the integral reduction algorithms developed for the all-massless four-point topologies [29, 30, 28, 27, 53], in order to reduce the loop integrals to a basis of master integrals. To put the integrands into a form suitable for applying the general reduction algorithms, spinor strings were converted to traces over $\gamma$ matrices, by multiplying and dividing by appropriate spinor inner products constructed from the external momenta. To illustrate the method consider the diagram depicted in fig. 1. The numerator of the integrand is

$$\langle 2^- | \gamma^\mu \not{k}_3 \gamma^\nu \not{k}_2 \gamma^\rho \not{k}_1 \gamma_\mu | 1^- \rangle \langle 4^+ | \gamma_\rho \not{k}_4 \gamma_\nu | 3^+ \rangle \,. \tag{5.1}$$

We multiply and divide this by

$$\langle 1^- | 4^+ \rangle \langle 3^+ | 2^- \rangle = \langle 14 \rangle [32] = \frac{\langle 14 \rangle}{\langle 23 \rangle} t \,, \tag{5.2}$$

so that the numerator can be rewritten as,

$$\begin{aligned}
&\frac{\langle 23 \rangle}{\langle 14 \rangle} \frac{1}{t} \langle 2^- | \gamma^\mu \not{k}_3 \gamma^\nu \not{k}_2 \gamma^\rho \not{k}_1 \gamma_\mu | 1^- \rangle \langle 1^- | 4^+ \rangle \langle 4^+ | \gamma_\rho \not{k}_4 \gamma_\nu | 3^+ \rangle \langle 3^+ | 2^- \rangle = \\
&\quad = \frac{\langle 23 \rangle}{\langle 14 \rangle} \frac{1}{t} \sum_{\text{spins}} \langle 2 | P_+ \gamma^\mu \not{k}_3 \gamma^\nu \not{k}_2 \gamma^\rho \not{k}_1 \gamma_\mu P_- | 1 \rangle \langle 1 | 4 \rangle \langle 4 | P_- \gamma_\rho \not{k}_4 \gamma_\nu P_+ | 3 \rangle \langle 3 | 2 \rangle \\
&\quad = \frac{\langle 23 \rangle}{\langle 14 \rangle} \frac{1}{t} \mathrm{Tr}\Big( \not{p}_2 P_+ \gamma^\mu \not{k}_3 \gamma^\nu \not{k}_2 \gamma^\rho \not{k}_1 \gamma_\mu P_- \not{p}_1 \not{p}_4 P_- \gamma_\rho \not{k}_4 \gamma_\nu P_+ \not{p}_3 \Big) \\
&\quad = \frac{\langle 23 \rangle}{\langle 14 \rangle} \frac{1}{t} \mathrm{Tr}\Big( \not{p}_2 \gamma^\mu \not{k}_3 \gamma^\nu \not{k}_2 \gamma^\rho \not{k}_1 \gamma_\mu \not{p}_1 \not{p}_4 P_- \gamma_\rho \not{k}_4 \gamma_\nu P_+ \not{p}_3 \Big) \,,
\end{aligned} \tag{5.3}$$

where $p_i$ is the momentum of external leg $i$ and we have introduced the helicity projectors $P_+$ and $P_-$, given by

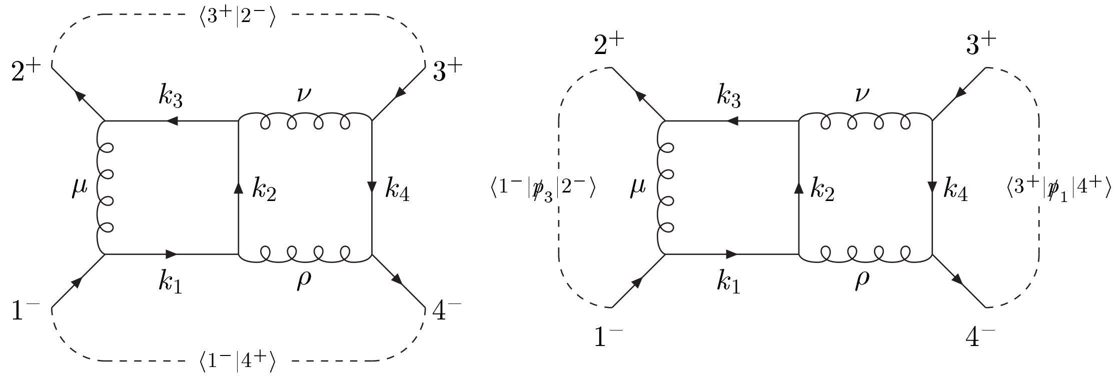
$$P_+ = \frac{1}{2}(1+\gamma_5) \,, \qquad P_- = \frac{1}{2}(1-\gamma_5) \,, \tag{5.4}$$

in order to reduce the sum over spins to the original helicity configuration.

To evaluate the projectors we use the 't Hooft-Veltman prescription [52] for $\gamma_5$,

$$\{\gamma_5, \gamma_\mu^{(4)}\} = 0 \,, \qquad [\gamma_5, \gamma_\mu^{(-2\epsilon)}] = 0 \,, \tag{5.5}$$

where the notation "(4)" and "$(-2\epsilon)$" is used to indicate whether the Lorentz index $\mu$ lies the four-dimensional or $(-2\epsilon)$-dimensional subspaces, respectively. (Another prescription is to take $\gamma_5$ to anti-commute [55] with all components, but this has the unwanted side-effect of ruining the smooth connection of the 't Hooft-Veltman and FDH scheme as a function




(a) (b)

**Figure 1:** An example of a two-loop diagram showing two different ways of "closing" the fermion lines. The dashed lines represent the spinor inner products inserted in order to form a trace.

$\delta_R$.) Using eq. (5.5) we move one of the projectors in the last line of eq. (5.3) until it hits the other, giving,

$$\begin{aligned}
\frac{\langle 23\rangle}{\langle 14\rangle}\frac{1}{t}\Big[&\mathrm{Tr}\Big(\not{p}_2\gamma^\mu \not{k}_3\gamma^\nu \not{k}_2\gamma^\rho \not{k}_1\gamma_\mu \not{p}_1\not{p}_4\gamma_\rho^{(4)}\not{k}_4^{(4)}\gamma_\nu^{(4)}P_+\not{p}_3\Big)+\\
&+\mathrm{Tr}\Big(\not{p}_2\gamma^\mu \not{k}_3\gamma^\nu \not{k}_2\gamma^\rho \not{k}_1\gamma_\mu \not{p}_1\not{p}_4\gamma_\rho^{(-2\epsilon)}\not{k}_4^{(-2\epsilon)}\gamma_\nu^{(4)}P_+\not{p}_3\Big)+\\
&+\mathrm{Tr}\Big(\not{p}_2\gamma^\mu \not{k}_3\gamma^\nu \not{k}_2\gamma^\rho \not{k}_1\gamma_\mu \not{p}_1\not{p}_4\gamma_\rho^{(-2\epsilon)}\not{k}_4^{(4)}\gamma_\nu^{(-2\epsilon)}P_+\not{p}_3\Big)+\\
&+\mathrm{Tr}\Big(\not{p}_2\gamma^\mu \not{k}_3\gamma^\nu \not{k}_2\gamma^\rho \not{k}_1\gamma_\mu \not{p}_1\not{p}_4\gamma_\rho^{(4)}\not{k}_4^{(-2\epsilon)}\gamma_\nu^{(-2\epsilon)}P_+\not{p}_3\Big)\Big]\,. \qquad (5.6)
\end{aligned}$$

In the last equation, the $\gamma_5$ part of $P_+$ will produce terms containing a Levi-Civita tensor. Upon integration this tensor can appear contracted only with external momenta, and since we have only three independent ones, these terms must vanish. Therefore the $\gamma_5$ term from the $P_+$ in eq. (5.6) can be dropped here.

After expanding the traces over Dirac matrices using standard formulæ, we obtain dot products of momenta (external and/or internal) in our expressions, making possible the application of the reduction algorithms previously mentioned. In evaluating the trace we take the four-dimensional and $(-2\epsilon)$-dimensional subspaces to be orthogonal following the discussion of ref. [43]. The rules for evaluating integrands containing dot products of $(-2\epsilon)$ dimensional components of momenta may be found in ref. [18].

Alternatively, we could have also chosen to "close" the fermion lines in a different way, multiplying and dividing, for instance, by

$$\langle 1^-|\not{p}_3|2^-\rangle\langle 3^+|\not{p}_1|4^+\rangle = \langle 14\rangle[32]u\,. \qquad (5.7)$$

This is depicted in figure 1($b$). However, this would have given a product of two traces instead of just one, which would produce some terms containing a product of two Levi-Civita tensors, which would then need to be evaluated since they would not vanish under integration.

After all the tensor loop integrals in the amplitudes have been reduced to a linear combination of master integrals, the next step is to expand the master integrals in a

Laurent series in $\epsilon$, beginning at order $1/\epsilon^4$, using results from refs. [25, 26] and [28]–[30]. The results can be expressed solely in terms of ordinary polylogarithms [62, 63],

$$\mathrm{Li}_n(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^n} = \int_0^x \frac{dt}{t}\,\mathrm{Li}_{n-1}(t)\,, \tag{5.8}$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\ln(1-t)\,, \tag{5.9}$$

with $n = 2, 3, 4$. The analytic properties of the non-planar double box integrals appearing in the amplitudes are somewhat intricate [11, 26]; there is no Euclidean region in any of the three kinematic channels, $s$, $t$ or $u$. We do not attempt to give a crossing-symmetric representation, but instead quote all our results in the physical $s$-channel ($s > 0$; $t, u < 0$) for the $q\bar{q} \to \bar{Q}Q$, $q\bar{Q} \to q\bar{Q}$ and $qQ \to qQ$ kinematics, eqs. (2.1)–(2.3).

### 5.2 Finite remainders

The two-loop finite remainders are defined in eq. (3.2) and are color decomposed into $M_h^{(2),[c]\mathrm{fin}}$ in eq. (2.18). Their dependence on the renormalization scale $\mu$, the number of colors $N$, the number of fermion flavors $N_f$, and scheme label $\delta_R$ may be extracted as

$$\begin{aligned}
M_h^{(2),[c]\mathrm{fin}} = & \Big[-b_0^2\left(\ln\left(\frac{s}{\mu^2}\right)-i\pi\right)^2 - b_1\left(\ln\left(\frac{s}{\mu^2}\right)-i\pi\right) - \frac{1}{36}(C_A+6C_F)C_A(1-\delta_R)^2 + \\
& + \Big(4R_q + \frac{1}{9}(C_A+9C_F)Q_0 + b_0 Q_1^{(qq)}\left(\ln\left(\frac{s}{\mu^2}\right)-i\pi\right) - \\
& - \frac{1}{3}(C_A-6C_F)Q_0 i\pi\Big)(1-\delta_R)\Big] M_h^{(0),[c]} + \\
& + \Big[-2b_0\left(\ln\left(\frac{s}{\mu^2}\right)-i\pi\right) + Q_1^{(qq)}(1-\delta_R)\Big] M_h^{(1),[c]\mathrm{fin}} + \\
& + Q_0 M_h^{(1),[c]\,\epsilon,\delta_R}(1-\delta_R) + P_h^{[c]}\,,
\end{aligned} \tag{5.10}$$

where,

$$\begin{aligned}
P_h^{[1]} &= N^2 A_h^{[1]} + B_h^{[1]} + \frac{1}{N^2}C_h^{[1]} + N N_f D_h^{[1]} + \frac{N_f}{N}E_h^{[1]} + N_f^2 F_h^{[1]}\,, \\
P_h^{[2]} &= N G_h^{[2]} + \frac{1}{N}H_h^{[2]} + \frac{1}{N^3}I_h^{[2]} + N_f J_h^{[2]} + \frac{N_f}{N^2}K_h^{[2]} + \frac{N_f^2}{N}L_h^{[2]}\,.
\end{aligned} \tag{5.11}$$

The $\mu$-dependence is a consequence of renormalization group invariance.

The tree and one-loop functions, $M_h^{(0),[c]}$ and $M_h^{(1),[c]\mathrm{fin}}$, are given in eq. (2.24) and eqs. (4.14) and (4.15), while $b_0$ and $b_1$ are given in eq. (2.7). The following combinations of color constants also appear in eq. (5.10),

$$Q_0 = \frac{5}{6}C_A - C_F + \frac{1}{3}T_R N_f\,, \tag{5.12}$$

$$Q_1^{(qq)} = -\frac{1}{3}C_A + C_F\,, \tag{5.13}$$

$$R_q = -\frac{7}{48}C_A^2 + \left(\frac{\pi^2}{192}+\frac{617}{864}\right)C_A C_F - \left(\frac{\pi^2}{24}+\frac{1}{4}\right)C_F^2 - \frac{1}{16}C_F T_R N_f\,. \tag{5.14}$$

The constants $Q_0$ and $R_q$ also appeared in the two-loop finite remainders for the $q\bar{q} \to gg$ process, and $Q_1^{(qq)}$ is just twice the constant $Q_1^{(qg)}$ appearing in that case. The quantities $M_h^{(1),[c]\,\epsilon,\delta_R}$ are the $\delta_R$-dependent parts of the $\mathcal{O}(\epsilon^1)$ coefficients of the one-loop amplitude remainders, after subtracting the poles using eq. (3.1). The explicit values of $M_h^{(1),[c]\,\epsilon,\delta_R}$ are tabulated in appendix B. The coefficient functions $A, B, C, D, E, F, G, H, I, J, K$ depend only on the Mandelstam variables. In appendix A, we give the explicit forms for the independent finite remainder functions appearing in eq. (5.11).

We have compared our results for the independent two-loop finite remainder functions $M_h^{(2),[c]\mathrm{fin}}$ with corresponding results obtained recently by Glover ref. [23]. Our results agree with the corrected version of ref. [23], once a slightly different definition of $\boldsymbol{H}^{(2)}(\epsilon)$ in eq. (3.7) is accounted for. (Instead of dressing the $1/\epsilon$ pole with $(\mu^2/(-s))^{2\epsilon}$, as we do in eq. (3.7), in ref. [23] it is dressed with $(\mu^2/(-s))^{2\epsilon} + (\mu^2/(-t))^{2\epsilon} - (\mu^2/(-u))^{2\epsilon}$). We have also checked that the interference of the tree and two-loop helicity amplitudes, summed over helicities and colors, reproduces the results given in ref. [13].

## 6. Amplitudes in pure $\mathcal{N} = 1$ super-Yang-Mills theory

Supersymmetric gauge theories have a wide range of applications both for phenomenological and theoretical purposes. Here we present the amplitudes in $\mathcal{N} = 1$ pure super-Yang-Mills theory, obtained from the QCD ones by modifying the fermions to be in the adjoint representation and by altering their multiplicity to correspond to a single Majorana fermion circulating in the loops. (A Majorana fermion counts as half of a Dirac fermion.) The fermion in the $\mathcal{N} = 1$ pure super-Yang-Mills theory is the gluino superpartner of the gluon. Besides the inherent interest in supersymmetric theories, a useful consequence is that supersymmetry imposes a set of powerful constraints on the amplitudes which can be used to verify their correctness. The supersymmetry identities have been applied previously to the same one-loop amplitudes discussed here [60], but only through $\mathcal{O}(\epsilon^0)$, as needed in an NLO calculation. In this section we discuss the supersymmetry identities for gluino-gluino scattering up to two loops.

### 6.1 Supersymmetry ward identities

In refs. [18, 21, 43] it was shown that using the FDH scheme ($\delta_R = 0$) the following identities are satisfied through $\mathcal{O}(\epsilon^0)$ at two loops,

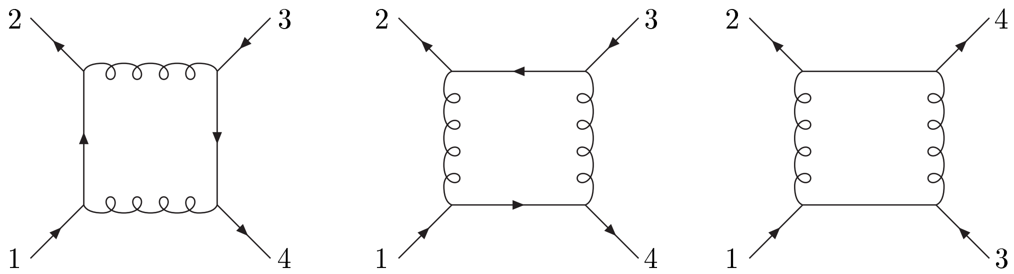
$$\mathcal{M}^{\mathrm{SUSY}}(g_1^{\pm}, g_2^{-}, g_3^{+}, g_4^{+}) = 0\,, \tag{6.1}$$

$$\mathcal{M}^{\mathrm{SUSY}}(\tilde{g}_1^{+}, \tilde{g}_2^{-}, g_3^{+}, g_4^{+}) = 0\,, \tag{6.2}$$

$$\mathcal{M}^{\mathrm{SUSY}}(\tilde{g}_1^{+}, \tilde{g}_2^{-}, g_3^{-}, g_4^{+}) = \frac{\langle 2\,3\rangle}{\langle 1\,3\rangle}\,\mathcal{M}^{\mathrm{SUSY}}(g_1^{+}, g_2^{-}, g_3^{-}, g_4^{+})\,, \tag{6.3}$$

where $g$ and $\tilde{g}$ denote a gluon and gluino, and the superscripts denote helicities. There is also an identity relating the four-gluino amplitudes to the four-gluon ones,

$$\mathcal{M}^{\mathrm{SUSY}}(\tilde{g}_1^{+}, \tilde{g}_2^{-}, \tilde{g}_3^{-}, \tilde{g}_4^{+}) = -\frac{\langle 2\,4\rangle}{\langle 1\,3\rangle}\,\mathcal{M}^{\mathrm{SUSY}}(g_1^{+}, g_2^{-}, g_3^{-}, g_4^{+})$$




(a) (b) (c)

**Figure 2:** Three contributions for Majorana fermions. The assignment of fermion arrows is inconsistent since there is no distinction between particles and antiparticles.

$$= -\frac{\langle 2\,4\rangle}{\langle 2\,3\rangle}\mathcal{M}^{\rm SUSY}(\tilde{g}_1^+, \tilde{g}_2^-, g_3^-, g_4^+) \qquad (6.4)$$

which we discuss here. In ref. [60] this was shown to be valid at one loop through $\mathcal{O}(\epsilon^0)$, as required in an NLO calculation.

### 6.2 Ambiguities in $D$-dimensional Dirac algebra

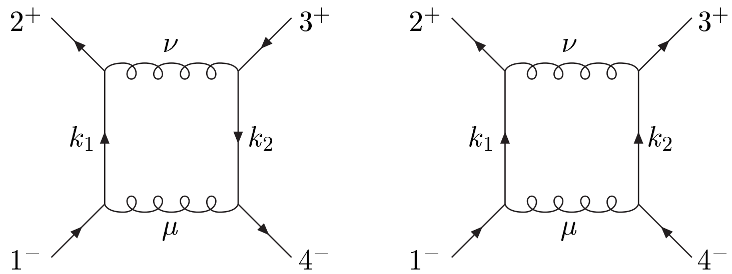
Before presenting the results, we first describe ambiguities that affect the amplitudes. The methods we use rely on continuing the Dirac algebra appearing in the spinor inner products away from four dimensions. This continuation is of course not unique, with a variety of schemes such as the 't Hooft-Veltman or FDH scheme for doing so. However, even within each of these schemes there can be further ambiguities. In particular, the charge conjugation properties of inner products of helicity states are not well defined as one moves away from four dimensions. In four dimensions we have the identity,

$$\langle p_1^\pm|\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}|p_2^\pm\rangle = \langle p_2^\mp|\gamma_{\mu_{2n+1}}\cdots\gamma_{\mu_1}|p_1^\mp\rangle\,, \qquad (6.5)$$

which is useful when performing Fierz rearrangements of the spinor products. It can also be used to construct simpler Dirac traces to evaluate. But can we use this identity when we continue the Lorentz indices of the Dirac matrices away from four dimensions?

For the case of Dirac fermions such as the quarks of QCD, we can avoid answering this question by imposing the rule that eq. (6.5) should not be used in any manipulations. Indeed, in performing the QCD calculations described in the previous section, we did not use this equation. However, with Majorana fermions, as appear in the $\mathcal{N}=1$ theory, there is no distinction between particles and antiparticles. In particular, one cannot consistently enforce a global requirement that a given fermion be treated as a particle and no global assignment of fermion arrows is consistent for all diagrams [64]. For example, in figure 2 if we chose legs 1 and 4 to be particles and legs 2 and 3 to be antiparticles, this would be inconsistent with diagram figure 2($c$) which is also a perfectly valid diagram for Majorana fermions.

Consider, for example, the box diagrams of figure 3. With Majorana fermions it is not clear which of the two diagrams should be used. The two diagrams differ by an application of the identity (6.5) and have identical values in four dimensions, up to the overall sign.




(a) (b)

**Figure 3:** A One-loop box diagram, and its "equivalent" where one fermion line has been charge-conjugated.

In order to calculate these diagrams we can multiply and divide the integrand of diagram $(a)$ by $\langle 1^-|4^+\rangle\langle 3^+|2^-\rangle$ and the integrand of diagram $(b)$ by $\langle 1^-|\not{p}_2|3^-\rangle\langle 4^-|\not{p}_1|2^-\rangle$. After performing the Dirac algebra, it is straightforward to show that the integrands of these two diagrams differ by a term proportional to $\lambda^2$, where $\lambda$ is the $(-2\epsilon)$-dimensional part of the loop momentum. Since the one-loop box diagram with a single $\lambda^2$ in the numerator is of $\mathcal{O}(\epsilon)$, this ambiguity is irrelevant at NLO. However, at two loops this ambiguity does enter, as one might expect by observing that $\mathcal{O}(\epsilon)$ terms at one loop contribute to $\mathcal{O}(1/\epsilon)$ terms at two loops in Catani's formula (3.2). With only a single or no fermion pair the results are unaffected by applying eq. (6.5). This ambiguity therefore does not appear in the $gggg$ or $\tilde{g}\tilde{g}gg$ amplitudes.

A related issue is the lack of helicity conservation at a fermion vertex with the 't Hooft-Veltman prescription [52] for $\gamma_5$ (5.5). Since the $(-2\epsilon)$-dimensional part of $\gamma$-matrices commute, helicity violating terms of the form $\langle a^-|\gamma^{(-2\epsilon)}|b^+\rangle = \langle a|P_+\gamma^{(-2\epsilon)}P_+|b\rangle$ do in general contribute. At tree level the helicity violating contributions vanish in the FDH scheme. However, they do contribute at loop level because $\mathcal{O}(\epsilon)$ terms are generated which interfere with the divergences. This is a reflection of the well known violation of the chiral Ward identity when using the 't Hooft-Veltman $\gamma_5$ prescription. Because of their connection to the divergences, it is not surprising that the fermion helicity violating terms can all be absorbed into Catani's formula for two-loop divergences, as confirmed by our calculations, and hence do not affect physical quantities. It is therefore perfectly consistent to drop these contributions (as we have done in QCD), if this is done systematically throughout the calculation. In $\mathcal{N} = 1$ super-Yang-Mills theory, although it is again consistent to drop these contributions, they do affect supersymmetry Ward identities.

One can choose various prescriptions for resolving these ambiguities, but it is not a priori clear which will allow the amplitudes to satisfy the supersymmetry identities. Whether or not one keeps fermion helicity violating contributions and the choices for assigning fermion arrows alters the value of the full amplitudes. Effectively, these ambiguities mean that the supersymmetry Ward identities will not hold unless an additional prescription is imposed. As we discuss below it is possible to choose the ambiguous terms so that the supersymmetry identities are satisfied.

However, a more straightforward way to deal with these ambiguities is to use the fact that they are all tied to the divergences controlled by Catani's formula. Our explicit calculations verify that the finite remainders after subtracting the divergences using Catani's formula are completely free of these ambiguities, as long as the tree, one-loop and two-loop amplitudes are treated uniformly with the same prescriptions. That is, we obtain the same finite remainders whether or not we apply eq. (6.5) to one of the fermion lines, or whether we keep or drop contributions that violate fermion helicity conservation. Moreover, as we discuss below, in the FDH scheme the finite remainders satisfy the expected supersymmetry identities without any additional prescriptions.

### 6.3 Color and infrared structure

Because the four gluinos are identical particles, and are Majorana, if we enforce helicity conservation, there are only two independent processes,

$$h=1: \quad \tilde{g}(p_1,+)+\tilde{g}(p_2,-) \to \tilde{g}(p_3,-)+\tilde{g}(p_4,+)\,, \tag{6.6}$$

$$h=2: \quad \tilde{g}(p_1,+)+\tilde{g}(p_2,+) \to \tilde{g}(p_3,+)+\tilde{g}(p_4,+)\,. \tag{6.7}$$

We ignore the helicity violating processes, *e.g.*,

$$h=3: \quad \tilde{g}(p_1,+)+\tilde{g}(p_2,-) \to \tilde{g}(p_3,+)+\tilde{g}(p_4,+)\,, \tag{6.8}$$

$$h=4: \quad \tilde{g}(p_1,-)+\tilde{g}(p_2,-) \to \tilde{g}(p_3,+)+\tilde{g}(p_4,+)\,, \tag{6.9}$$

since, as mentioned above, and confirmed by explicit computation, they can be absorbed entirely into Catani's formula for divergences.

Since the gluinos are in the adjoint representation we use the same color basis as used for the four-gluon helicity amplitudes [18]

$$\tilde{\mathcal{M}}_h^{(L)} = \tilde{S}_h \times \sum_{c=1}^{9} \mathrm{Tr}^{[c]} \times \tilde{M}_h^{(L),[c]}\,, \tag{6.10}$$

where

$$\begin{aligned}
\mathrm{Tr}^{[1]} &= \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,, & \mathrm{Tr}^{[2]} &= \mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\,,\\
\mathrm{Tr}^{[3]} &= \mathrm{tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\,, & \mathrm{Tr}^{[4]} &= \mathrm{tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})\,,\\
\mathrm{Tr}^{[5]} &= \mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})\,, & \mathrm{Tr}^{[6]} &= \mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})\,,\\
\mathrm{Tr}^{[7]} &= \mathrm{tr}(T^{a_1}T^{a_2})\,\mathrm{tr}(T^{a_3}T^{a_4})\,, & \mathrm{Tr}^{[8]} &= \mathrm{tr}(T^{a_1}T^{a_3})\,\mathrm{tr}(T^{a_2}T^{a_4})\,,\\
\mathrm{Tr}^{[9]} &= \mathrm{tr}(T^{a_1}T^{a_4})\,\mathrm{tr}(T^{a_2}T^{a_3})\,. & &
\end{aligned} \tag{6.11}$$

A reflection identity implies that the $c = 4,5,6$ coefficients are equal to the $c = 3,2,1$ coefficients (respectively), so there are really only six different coefficients for each $h$, namely $\tilde{M}_h^{(L),[c]}$, $c = 1,2,3,7,8,9$. The corresponding spinor factors are

$$\tilde{S}_1 = S_1 = -i\frac{\langle 3\,1\rangle}{\langle 4\,2\rangle}\,, \qquad \tilde{S}_2 = S_5 = -i\frac{\langle 2\,1\rangle}{\langle 4\,3\rangle}\,. \tag{6.12}$$

The infrared divergence structure is similar to that of gluon-gluon and gluino-gluon scattering amplitudes [18, 21]. For the case of $\mathcal{N}=1$ pure super-Yang-Mills theory, in the basis (6.11) the matrix $\boldsymbol{I}^{(1)}$ is [15, 18, 21]

$$\tilde{\boldsymbol{I}}^{(1)}(\epsilon) = -\frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{\tilde{b}_0}{N\epsilon}\right)\times \tag{6.13}$$
$$\times\begin{pmatrix}
N(\mathtt{S}+\mathtt{T}) & 0 & 0 & 0 & 0 & 0 & (\mathtt{T}-\mathtt{U}) & 0 & (\mathtt{S}-\mathtt{U}) \\
0 & N(\mathtt{S}+\mathtt{U}) & 0 & 0 & 0 & 0 & (\mathtt{U}-\mathtt{T}) & (\mathtt{S}-\mathtt{T}) & 0 \\
0 & 0 & N(\mathtt{T}+\mathtt{U}) & 0 & 0 & 0 & 0 & (\mathtt{T}-\mathtt{S}) & (\mathtt{U}-\mathtt{S}) \\
0 & 0 & 0 & N(\mathtt{T}+\mathtt{U}) & 0 & 0 & 0 & (\mathtt{T}-\mathtt{S}) & (\mathtt{U}-\mathtt{S}) \\
0 & 0 & 0 & 0 & N(\mathtt{S}+\mathtt{U}) & 0 & (\mathtt{U}-\mathtt{T}) & (\mathtt{S}-\mathtt{T}) & 0 \\
0 & 0 & 0 & 0 & 0 & N(\mathtt{S}+\mathtt{T}) & (\mathtt{T}-\mathtt{U}) & 0 & (\mathtt{S}-\mathtt{U}) \\
(\mathtt{S}-\mathtt{U}) & (\mathtt{S}-\mathtt{T}) & 0 & 0 & (\mathtt{S}-\mathtt{T}) & (\mathtt{S}-\mathtt{U}) & 2N\mathtt{S} & 0 & 0 \\
0 & (\mathtt{U}-\mathtt{T}) & (\mathtt{U}-\mathtt{S}) & (\mathtt{U}-\mathtt{S}) & (\mathtt{U}-\mathtt{T}) & 0 & 0 & 2N\mathtt{U} & 0 \\
(\mathtt{T}-\mathtt{U}) & 0 & (\mathtt{T}-\mathtt{S}) & (\mathtt{T}-\mathtt{S}) & 0 & (\mathtt{T}-\mathtt{U}) & 0 & 0 & 2N\mathtt{T}
\end{pmatrix}$$

where $\mathtt{S}$, $\mathtt{T}$ and $\mathtt{U}$ are defined in eq. (3.11). For $\mathcal{N}=1$ super-Yang-Mills theory the first two coefficients of the $\beta$-function are

$$\tilde{b}_0 = \frac{3}{2}C_A\,, \qquad \tilde{b}_1 = \frac{3}{2}C_A^2\,. \tag{6.14}$$

The $\boldsymbol{I}^{(2)}$ operator for super-Yang-Mills theory is

$$\begin{aligned}
\tilde{\boldsymbol{I}}^{(2)}_{\mathrm{FDH}}(\epsilon,\mu;\{p\}) = & -\frac{1}{2}\tilde{\boldsymbol{I}}^{(1)}(\epsilon,\mu;\{p\})\left(\tilde{\boldsymbol{I}}^{(1)}(\epsilon,\mu;\{p\})+\frac{2\tilde{b}_0}{\epsilon}\right)+ \\
& +\frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\tilde{b}_0}{\epsilon}+K^{\mathrm{SYM}}_{\mathrm{FDH}}\right)\tilde{\boldsymbol{I}}^{(1)}(2\epsilon,\mu;\{p\})+ \\
& +\tilde{\boldsymbol{H}}^{(2)}_{\mathrm{FDH}}(\epsilon,\mu;\{p\})\,,
\end{aligned} \tag{6.15}$$

where

$$K^{\mathrm{SYM}}_{\mathrm{FDH}} = \left(3-\frac{\pi^2}{6}-\frac{4}{9}\epsilon\right)C_A\,, \tag{6.16}$$

$$\tilde{\boldsymbol{H}}^{(2)}_{\mathrm{FDH}}(\epsilon,\mu;\{p\}) = \frac{e^{-\epsilon\psi(1)}}{4\epsilon\,\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^{2\epsilon}\left(4(H_g^{(2)})^{\mathrm{SYM}}_{\mathrm{FDH}}\,\mathbf{1}+\hat{\boldsymbol{H}}^{(2)}\right), \tag{6.17}$$

and

$$\left(H_g^{(2)}\right)^{\mathrm{SYM}}_{\mathrm{FDH}} = \left(H_{\tilde{g}}^{(2)}\right)^{\mathrm{SYM}}_{\mathrm{FDH}} = \left(\frac{\zeta_3}{2}+\frac{\pi^2}{16}-\frac{2}{9}\right)C_A^2\,. \tag{6.18}$$

The equality of $(H_g^{(2)})^{\mathrm{SYM}}_{\mathrm{FDH}}$ and $(H_{\tilde{g}}^{(2)})^{\mathrm{SYM}}_{\mathrm{FDH}}$ is a consequence of supersymmetry. Equations (6.16)–(6.18) are obtained from the QCD formulas, eqs. (3.7)–(3.8), by the replacements $\delta_R \to 0$, $C_F \to C_A$ and $T_R N_F \to C_A/2$ for converting to a single adjoint fermion in the FDH scheme. The operator $\hat{\boldsymbol{H}}^{(2)}$ defined in eq. (3.9) does not explicitly depend on the fermion representation.

The tree amplitudes in this color basis in the FDH scheme are,

$$
\begin{aligned}
&\tilde{M}_1^{(0),[1]} = \frac{u}{s} + \frac{u}{t}\,, &&\tilde{M}_1^{(0),[2]} = -\frac{u}{s}\,, &&\tilde{M}_1^{(0),[3]} = -\frac{u}{t}\,,\\
&\tilde{M}_2^{(0),[1]} = \frac{s}{t}\,, &&\tilde{M}_2^{(0),[2]} = \frac{s}{u}\,, &&\tilde{M}_2^{(0),[3]} = -\frac{s}{u} - \frac{s}{t}\,,\\
&\tilde{M}_3^{(0),[1]} = 0\,, &&\tilde{M}_3^{(0),[2]} = 0\,, &&\tilde{M}_3^{(0),[3]} = 0\,,\\
&\tilde{M}_h^{(0),[3-i]} = \tilde{M}_h^{(0),[4+i]}\,, \qquad h = 1,2,3, \qquad i = 0,1,2,\\
&\tilde{M}_h^{(0),[c]} = 0\,, \qquad c = 7,8,9, \qquad h = 1,2,3\,,
\end{aligned}
\tag{6.19}
$$

and are free of the ambiguities described above.

### 6.4 One-loop amplitudes in pure $\mathcal{N} = 1$ super-Yang-Mills theory

We now present the results for one-loop four-gluino scattering in a format valid to all orders in $\epsilon$. However, because of the ambiguity discussed above the answer depends on the precise steps used to calculate the expression.

We can, for example, form the $h = 1$ gluino amplitude from the $h = 3$ (with traces formed by multiplying and dividing by $[1\,2]\,\langle 4\,3\rangle$) amplitude (2.13) with adjoint representation quarks and $N_f = 1/2$ summed with its $3 \leftrightarrow 4$ interchange. This gives, for the unrenormalized coefficients of the three color structures $\mathrm{Tr}^{[1]}$, $\mathrm{Tr}^{[2]}$ and $\mathrm{Tr}^{[3]}$,

$$
\begin{aligned}
M_1^{(1),[1]} &= \frac{N}{2} u\Bigg[-\frac{\epsilon(4s+t)+2u}{(2\epsilon-1)t}\mathrm{Tri}^{(4)}(s) - \frac{\epsilon(4t+s)+2u}{(2\epsilon-1)s}\mathrm{Tri}^{(4)}(t) + \\
&\qquad + \left((4\epsilon-2)\left(\frac{s}{t}+\frac{t}{s}\right) + (5\epsilon-1)\right)\mathrm{Box}^{(6)}(s,t)\Bigg]\,, && (6.20)\\
M_1^{(1),[2]} &= Nu\left[\frac{\epsilon-2}{2(2\epsilon-1)}\mathrm{Tri}^{(4)}(s) + \frac{u}{s}\mathrm{Tri}^{(4)}(u) - \left(\epsilon\frac{s+2u}{s} + \frac{t}{s}\right)\mathrm{Box}^{(6)}(u,s)\right], && (6.21)\\
M_1^{(1),[3]} &= Nu\left[\frac{u}{t}\mathrm{Tri}^{(4)}(u) + \frac{\epsilon-2}{2(2\epsilon-1)}\mathrm{Tri}^{(4)}(t) - \left(\epsilon\frac{t+2u}{t} + \frac{s}{t}\right)\mathrm{Box}^{(6)}(u,t)\right], && (6.22)
\end{aligned}
$$

where we have taken the FDH scheme ($\delta_R = 0$). The renormalization is trivially performed by subtracting the quantity $\tilde{b}_0\tilde{M}_h^{(0),[c]}/\epsilon$ from each, where $\tilde{b}_0$ is given in eq. (6.14). The ambiguities are all proportional to $\epsilon\,\mathrm{Box}^{(6)}$ and hence are of $\mathcal{O}(\epsilon)$.

Other choices are also possible. For example, if include a fermion helicity violating term $q^+\bar{q}^+ \to Q^+\bar{Q}^+$ (forming traces by multiplying and dividing by $\langle 4^-|\not{k}_3|1^-\rangle\langle 3^-|\not{k}_1|2^-\rangle$) in constructing the $h = 2$ gluino amplitude, this shifts eq. (6.21) by

$$
\delta M_1^{(1),[2]} = \frac{1}{2} N u\,\epsilon\,\mathrm{Box}^{(6)}(u,s)\,, \tag{6.23}
$$

leaving the coefficients of the first and third color structures unshifted. Similarly, adding in the fermion helicity violating contribution $q^+\bar{q}^+ \to \bar{Q}^+Q^+$ shifts eq. (6.22) by

$$
\delta M_1^{(1),[3]} = \frac{1}{2} N u\,\epsilon\,\mathrm{Box}^{(6)}(u,t)\,. \tag{6.24}
$$

For any of the above contributions, we can swap one of the particles with its antiparticle before converting them to gluinos, which again alters their values by $\mathcal{O}(\epsilon)$ terms proportional to $\epsilon\,\mathrm{Box}^{(6)}$.

The remaining color coefficients, up to $\mathcal{O}(\epsilon)$ ambiguous terms, are given in terms of the previous ones by,

$$\begin{aligned}
\tilde{M}_h^{(1),[3-i]}(s,t,u) &= \tilde{M}_h^{(1),[4+i]}(s,t,u) \qquad i=0,1,2,\ , \\
\tilde{M}_h^{(1),[7]}(s,t,u) &= \frac{2}{N}\Big(\tilde{M}_h^{(1),[3]}(s,t,u)+\tilde{M}_h^{(1),[2]}(s,t,u)+\tilde{M}_h^{(1),[1]}(s,t,u)\Big)\,, \\
\tilde{M}_h^{(1),[8]}(s,t,u) &= \tilde{M}_h^{(1),[9]}(s,t,u)=\tilde{M}_h^{(1),[7]}(s,t,u)\,.
\end{aligned} \tag{6.25}$$

The $h=2$ amplitudes are related to the $h=1$ through the following relations,

$$\begin{aligned}
\tilde{M}_2^{(1),[1]}(s,t,u) &= -\tilde{M}_1^{(1),[2]}(t,u,s)\,, \\
\tilde{M}_2^{(1),[2]}(s,t,u) &= -\tilde{M}_1^{(1),[3]}(t,u,s)\,, \\
\tilde{M}_2^{(1),[3]}(s,t,u) &= -\tilde{M}_1^{(1),[1]}(t,u,s)\,, \\
\tilde{M}_2^{(1),[7]}(s,t,u) &= -\tilde{M}_1^{(1),[7]}(t,u,s)\,.
\end{aligned} \tag{6.26}$$

To check the supersymmetry identity (6.4) we compare these results to the corresponding ones for gluon-gluon and gluino-gluon scattering given in refs. [18, 21]. We see that eqs. (6.20), (6.21) and (6.22) match eqs. (5.28), (5.29) and (5.30) in ref. [21] respectively, up to an overall normalization factor of $-st/u^2$ (which is similar to the same factor in eq. (5.39) of ref. [21]), except for terms proportional to $\epsilon\,\mathrm{Box}^{(6)}$ in $\tilde{M}_2^{(1),[2]}$ and $\tilde{M}_2^{(1),[3]}$. (Note that in ref. [21] the amplitudes include the renormalization subtraction, not explicitly included here.) Similarly it matches the corresponding equations in section 3.1 of ref. [18] for gluon-gluon scattering, again up to the terms proportional to $\epsilon\mathrm{Box}^{(6)}$. Since $\mathrm{Box}^{(6)}(s,u)$ is finite, the one-loop terms that violate the supersymmetry Ward identity are of $\mathcal{O}(\epsilon)$. These terms are precisely the ambiguous terms described above. We can alter such terms by modifying the prescription. If, for example, we include in the shifts (6.23) and (6.24) arising from fermion helicity violating terms we find that the supersymmetry identities are then satisfied to all orders in $\epsilon$. Of course, this is not completely satisfactory because of the ad hoc nature of such choices.

### 6.5 Two-loop amplitudes in $\mathcal{N}=1$ super-Yang-Mills theory

The dependence of the identity (6.4) beyond $\mathcal{O}(\epsilon^0)$ on ambiguous terms at one loop foretells a similar dependence at two loops except this time at $\mathcal{O}(1/\epsilon)$. However, after subtracting these divergences, via Catani's formula, we find that eq. (6.4) is indeed satisfied for the finite terms which are independent of the ambiguous terms and any prescriptions chosen for fixing them. However, some attention is required to ensure that precisely the same prescriptions are applied to tree level and one loop as applied to two loops. Otherwise, there would be a mismatch between the Catani subtraction terms and the two-loop divergences leaving behind dependence on the ambiguous terms.



JHEP09(2004)039

Our results in the FDH scheme are

$$\begin{aligned}
M_h^{(2),\text{SYM},[c]\text{fin}} &= -\Big[(\tilde{b}_0)^2 \left(\ln\left(\frac{s}{\mu^2}\right) - i\pi\right)^2 + \tilde{b}_1 \left(\ln\left(\frac{s}{\mu^2}\right) - i\pi\right)\Big] M_h^{(0),[c]} - \\
&\quad - 2\tilde{b}_0 \left(\ln\left(\frac{s}{\mu^2}\right) - i\pi\right) M_h^{(1),\text{SYM},[c]\text{fin}} + N^2 A_h^{\text{SYM},[c]} + B_h^{\text{SYM},[c]}\,, \\
&\qquad c = 1, 2, 3,
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
M_h^{(2),\text{SYM},[c]\text{fin}} &= -2\tilde{b}_0 \left(\ln\left(\frac{s}{\mu^2}\right) - i\pi\right) M_h^{(1),\text{SYM},[c]\text{fin}} + N\, G_h^{\text{SYM},[c]}\,, \\
&\qquad c = 7, 8, 9.
\end{aligned} \tag{6.28}$$

Because the adjoint color indices of the gluino fields are identical to those of gluons, and only structure constants $f^{abc}$ appear in the two-loop Feynman diagrams, the color coefficients for two-gluino two-gluon scattering obey the same group theory relations identified for $gg \to gg$ in ref. [18],

$$G_h^{\text{SYM},[7]} = 2\Big(A_h^{\text{SYM},[1]} + A_h^{\text{SYM},[2]} + A_h^{\text{SYM},[3]}\Big) - B_h^{\text{SYM},[3]}\,, \tag{6.29}$$

$$G_h^{\text{SYM},[8]} = 2\Big(A_h^{\text{SYM},[1]} + A_h^{\text{SYM},[2]} + A_h^{\text{SYM},[3]}\Big) - B_h^{\text{SYM},[1]}\,, \tag{6.30}$$

$$G_h^{\text{SYM},[9]} = 2\Big(A_h^{\text{SYM},[1]} + A_h^{\text{SYM},[2]} + A_h^{\text{SYM},[3]}\Big) - B_h^{\text{SYM},[2]}\,, \tag{6.31}$$

and

$$B_h^{\text{SYM},[3]} = -B_h^{\text{SYM},[1]} - B_h^{\text{SYM},[2]}\,. \tag{6.32}$$

We have verified that the finite remainder functions match those of the pure gluon case, i.e.

$$X_1^{\text{SYM},[c]} = X_{-+-+}^{\text{SYM},[c]}\,, \tag{6.33}$$

$$X_2^{\text{SYM},[c]} = -X_{--++}^{\text{SYM},[c]}\,, \tag{6.34}$$

where $X \in \{A, B, G\}$. The functions $X_{-+-+}^{\text{SYM},[c]}$ and $X_{--++}^{\text{SYM},[c]}$ for $gg \to gg$ scattering in pure $\mathcal{N} = 1$ super-Yang-Mills theory are given in ref. [18]. (Note that in that reference an all outgoing definition of helicity is used.) The relations (6.33)–(6.34) are precisely equivalent to the content of the supersymmetry Ward identities [42], after removing overall factors and the divergent terms, and extracting the $N$ and $\mu$ dependence. As discussed in ref. [21] this also matches the corresponding functions for the $\tilde{g}\tilde{g} \to gg$ amplitudes. This demonstrates that the supersymmetry identity (6.4) holds for the finite remainders at two loops. In the HV scheme, as expected, it does not hold because of the mismatch of fermionic and bosonic states.

## 7. Conclusions

In this paper we presented the two-loop helicity amplitudes for quark-quark and anti-quark-quark scattering in QCD and gluino-gluino scattering in $\mathcal{N} = 1$ super-Yang-Mills theory. These amplitudes retain the full information on external colors and helicities. We verified that the interference of our two-loop amplitudes with the tree-level amplitudes, summed over all external colors and helicities, and converted to the CDR scheme, are in complete agreement with the results of ref. [13]. We also found complete agreement with recently published results by Glover [23], after correction of minor errors in the original version of that article.

We also discuss ambiguities in defining the amplitudes, related to the continutation of $\gamma_5$ or charge conjugation identities away from four dimensions. As confirmed by our explicit calculations these ambiguities drop out of final physical results, since the ambiguous contributions can be absorbed into infrared singularities that cancel from physical quantities. In particular, there is no physical content to contributions that violate helicity conservation of a given massless fermion. However, some attention needs to be paid so that the different loop orders are computed consistently with the same set of prescriptions throughout. Otherwise, the cancellation of the ambiguous unphysical terms would not be complete.

In ref. [21] it was shown that the supersymmetry Ward identities relating the $\tilde{g}\tilde{g} \to gg$ to the $gg \to gg$ processes are satisfied in the FDH scheme at two loops, for both infrared divergent and finite parts. For the gluino-gluino scattering amplitudes discussed in this paper, unless the ambiguities entering in the amplitude are carefully adjusted, the supersymmetry identities will not hold starting at $\mathcal{O}(1/\epsilon)$. In any case, after subtracting the divergences using Catani's formula [31], the finite remainders are free of these ambiguities and satisfy the expected supersymmetry identities in the FDH scheme.

So far, the new $2 \to 2$ amplitudes amplitudes have been implemented in a handful of new phenomenological studies [17, 65]. Once general algorithms for dealing with infrared divergent phase space integrations at next-to-next-leading-order are completed [66], many more phenomenological applications will follow. These applications would include the implementation of the two-loop four-quark amplitudes of this paper, or those of refs. [13, 23], as ingredients in a numerical program for computing dijet production cross sections at hadron colliders at NNLO in QCD. When this task is accomplished, the intrinsic precision on the QCD predictions should reach the few percent level.

## Acknowledgments

We thank Lance Dixon for key contributions at early stages of this paper and Babis Anastasiou, David Kosower and Henry Wong for helpful comments. We also thank Nigel Glover for communications regarding ref. [23]. A.D.F. thanks the Alexander von Humboldt Foundation for its support.

## A. Finite remainder functions for QCD

In this appendix, we present the explicit forms for the independent finite remainder functions for the processes $q\bar{q} \to \bar{Q}Q$, $q\bar{Q} \to q\bar{Q}$ and $qQ \to qQ$ in QCD, which appear in eq. (5.11).

For the helicity $h = 1$ configuration in eq. (2.11) and color factor $\mathrm{Tr}^{[1]}$ in eq. (2.19), the finite remainder functions are:

$$
\begin{aligned}
A_1^{[1]} = \bigg[ & (x-1)\mathrm{Li}_4(-x) - \left(xX + \frac{19}{6} + \frac{11}{6}x\right)\mathrm{Li}_3(-x) + \left(\frac{1}{24} + \frac{1}{12}x\right)X^4 + \\
& + \frac{1}{6}(11x+19)X\,\mathrm{Li}_2(-x) + \bigg(\left(\frac{85}{18} - \frac{5}{12}\pi^2\right)x - \frac{\pi^2}{3} + \frac{109}{18}\bigg)X^2 + \\
& + \bigg(\left(-\frac{1}{2}\zeta_3 - \frac{79}{27} + \frac{373}{144}\pi^2\right)x - \frac{62}{27} - 3\zeta_3 + \frac{59}{24}\pi^2\bigg)X + \\
& + \left(\frac{197}{72}\zeta_3 + \frac{23213}{5184} + \frac{113}{1440}\pi^4\right)x + \frac{233}{36}\zeta_3 + \frac{23213}{2592} + \frac{137}{720}\pi^4\bigg]\frac{x}{y} + \\
& + \bigg[-\frac{1}{2}\mathrm{Li}_4(-x) - \frac{4}{3}\mathrm{Li}_3(-x) + \left(\frac{1}{4}X^2 + \frac{4}{3}X\right)\mathrm{Li}_2(-x) + \frac{1}{48}X^4 - \frac{37}{72}X^3 + \\
& \quad + \left(\frac{28}{9} - \frac{\pi^2}{6}\right)X^2 + \left(\frac{17}{27} - \frac{3}{2}\zeta_3 + \frac{43}{48}\pi^2\right)X + \frac{269}{72}\zeta_3 + \frac{137}{1440}\pi^4 + \\
& \quad + \frac{23213}{5184}\bigg]\frac{1}{y} - \frac{1}{2}xX^2\mathrm{Li}_2(-x) + \frac{49}{36}xX^3 - \frac{\pi^2}{3}y\mathrm{Li}_2(-x) - \\
& - \left(\frac{11}{12}x + \frac{2}{3}\right)YX^2 + \frac{61}{36}\pi^2 + \frac{49}{18}\pi^2 x - \frac{\pi^2}{3}yYX + \frac{1}{6}yX^3Y + \\
& + i\pi\bigg\{\bigg[-x\mathrm{Li}_3(-x) + \left(\frac{11}{6}x + \frac{19}{6}\right)\mathrm{Li}_2(-x) + \left(\frac{1}{6} + \frac{1}{3}x\right)X^3 + \\
& \qquad + \frac{1}{9}(85x+109)X - \left(\frac{1}{2}\zeta_3 + \frac{79}{27}\right)x - 3\zeta_3 - \frac{62}{27}\bigg]\frac{x}{y} - xX\mathrm{Li}_2(-x) + \\
& \qquad + \bigg[\left(\frac{1}{2}X + \frac{4}{3}\right)\mathrm{Li}_2(-x) - \frac{3}{2}\zeta_3 + \frac{1}{12}X^3 + \frac{17}{27} - \frac{37}{24}X^2 + \frac{56}{9}X\bigg]\frac{1}{y} - \\
& \qquad - \left(\frac{11}{6}x + \frac{4}{3}\right)YX + \frac{49}{12}xX^2 + \left(\frac{1}{2}YX^2 - \frac{\pi^2}{6}X - \frac{19}{144}\pi^2\right)y\bigg\},
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
B_1^{[1]} = \bigg[ & -10\mathrm{Li}_4\left(-\frac{x}{y}\right) - (11+3x)\mathrm{Li}_4(-x) + 10\mathrm{Li}_4(-y) + \\
& + \left((2+3x)X + \frac{11}{3}x + 8Y + \frac{47}{6}\right)\mathrm{Li}_3(-x) - \left(\frac{11}{3}x + 4X + \frac{40}{3}\right)\mathrm{Li}_3(-y) - \\
& - \bigg(\left(\frac{47}{6} + 2Y + \frac{11}{3}x\right)X + \frac{1}{3}(11x+40)Y - \frac{5}{3}\pi^2 - \pi^2 x\bigg)\mathrm{Li}_2(-x) - \\
& - \left(\frac{1}{8} + \frac{1}{4}x\right)X^4 - \bigg(\left(\frac{1}{2}x + \frac{2}{3}\right)Y - \frac{125}{36}x - \frac{119}{36}\bigg)X^3 + \\
& + \bigg(\left(\frac{1}{4}x - \frac{3}{2}\right)Y^2 - \left(\frac{53}{12} + \frac{31}{12}x\right)Y + \left(\frac{3}{2}\pi^2 - \frac{617}{72}\right)x - \frac{227}{18} + \frac{5}{4}\pi^2\bigg)X^2 + \\
& + \frac{5}{3}Y^3X + \bigg(\left(\frac{9}{4} - \frac{2}{3}\pi^2\right)x + \frac{\pi^2}{3} + \frac{19}{4}\bigg)YX - \left(\frac{4}{27} + \frac{373}{72}\pi^2 - \zeta_3\right)xX - \\
& - \left(\frac{119}{27} - 6\zeta_3 + \frac{14}{3}\pi^2\right)X - \left(\frac{1}{6} - \frac{1}{8}x\right)Y^4 - \pi^2\left(\frac{1}{3}x + \frac{3}{2}\right)Y^2 -
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\left(\zeta_3+\frac{158}{27}\right)x+\frac{316}{27}+10\zeta_3\right)Y+\frac{19}{720}\pi^4-\frac{263}{36}\zeta_3-\frac{167}{72}\pi^2+\\
&+\left(\frac{30659}{1296}-\frac{443}{72}\zeta_3+\frac{79}{1440}\pi^4-\frac{55}{144}\pi^2\right)x+\frac{30659}{648}\Bigg]\frac{x}{y}+\frac{3}{2}xX^2\mathrm{Li}_2(-x)+\\
&+\Bigg[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-\frac{5}{2}\mathrm{Li}_4(-x)+2\mathrm{Li}_4(-y)+\left(Y+X+\frac{19}{6}\right)\mathrm{Li}_3(-x)-\\
&\quad-\left(\frac{29}{3}+2X\right)\mathrm{Li}_3(-y)-\left(\frac{3}{4}X^2+\left(\frac{19}{6}+Y\right)X-\frac{4}{3}\pi^2+\frac{29}{3}Y\right)\mathrm{Li}_2(-x)-\\
&\quad-\frac{1}{16}X^4-\left(\frac{1}{3}Y-\frac{89}{72}\right)X^3+\left(\frac{5}{8}\pi^2-\frac{403}{72}-\frac{41}{24}Y-\frac{3}{4}Y^2\right)X^2+\\
&\quad+\left(\frac{1}{3}Y^3+\left(\frac{\pi^2}{6}+\frac{5}{2}\right)Y-\frac{115}{27}-\frac{41}{24}\pi^2+3\zeta_3\right)X+\frac{1}{24}Y^4-\frac{\pi^2}{2}Y^2-\\
&\quad-\left(2\zeta_3+\frac{158}{27}\right)Y+\frac{30659}{1296}-\frac{31}{16}\pi^2-\frac{13}{288}\pi^4-\frac{11}{72}\zeta_3\Bigg]\frac{1}{y}+\\
&+\frac{1}{12}\Big(67+31x\Big)Y^2X-\pi^2\left(\frac{45}{8}+\frac{139}{24}x\right)Y-\left[\frac{71}{36}xY^3+\left(\frac{5}{2}-\frac{599}{72}x\right)Y^2\right]\frac{y}{x}+\\
&+i\pi\Bigg\{\Bigg[(3x+10)\mathrm{Li}_3(-x)-4\mathrm{Li}_3(-y)-\left(\frac{1}{2}+x\right)X^3+\left(\frac{113}{12}+\frac{29}{3}x\right)X^2-\\
&\qquad-\left((5+3x)X+\frac{22}{3}x+\frac{127}{6}+2Y\right)\mathrm{Li}_2(-x)-\left(1-\frac{1}{2}x\right)Y^2X-\\
&\qquad-\left(\left(\frac{71}{6}+\frac{20}{3}x\right)Y-\left(\frac{\pi^2}{3}-\frac{134}{9}\right)x+\frac{737}{36}-\frac{\pi^2}{2}\right)X+\left(\frac{11}{18}\pi^2-6\right)x+\\
&\qquad+\frac{53}{36}\pi^2-\left(\left(\frac{\pi^2}{3}-\frac{170}{9}\right)x+\frac{\pi^2}{3}-\frac{1189}{36}\right)Y-4\zeta_3-\frac{145}{9}\Bigg]\frac{x}{y}+\\
&\qquad+\Bigg[2\mathrm{Li}_3(-x)-2\mathrm{Li}_3(-y)-\left(\frac{77}{6}+\frac{5}{2}X+Y\right)\mathrm{Li}_2(-x)-\frac{1}{4}X^3+\frac{43}{12}X^2-\\
&\qquad\quad-\left(\frac{59}{12}Y-\frac{\pi^2}{4}+\frac{313}{36}+\frac{1}{2}Y^2\right)X-\left(\frac{\pi^2}{6}-\frac{329}{36}\right)Y+\\
&\qquad+\zeta_3-\frac{91}{9}+\frac{25}{36}\pi^2\Bigg]\frac{1}{y}-\left(YX^2-\frac{1}{2}Y^3+\frac{20}{3}Y^2\right)y-\frac{5}{xy}Y\Bigg\}, \qquad\qquad (\mathrm{A.2})
\end{aligned}
$$

$$
\begin{aligned}
C_1^{[1]}=&\Bigg[12\mathrm{Li}_4\left(-\frac{x}{y}\right)+(4x+20)\mathrm{Li}_4(-x)+(4+8x)\mathrm{Li}_4(-y)-\\
&-((4x+6)X+4+6Y)\mathrm{Li}_3(-x)+(12X-(16+8x)Y)\mathrm{Li}_3(-y)+\\
&+\left((6Y+4)X+\frac{4}{3}\pi^2x+\frac{2}{3}\pi^2\right)\mathrm{Li}_2(-x)+\left(\frac{7}{12}+\frac{1}{24}x\right)Y^4+\\
&+\left(\frac{1}{12}+\frac{5}{24}x\right)X^4+\left(\frac{1}{3}(1+2x)Y-\frac{9}{4}x-\frac{11}{6}\right)X^3-\frac{\pi^2}{2}xY^2-\\
&-\left(\left(\frac{3}{4}x-\frac{9}{2}\right)Y^2-\left(\frac{7}{2}+\frac{9}{4}x\right)Y+\left(\frac{5}{8}+\frac{11}{6}\pi^2\right)x+\frac{7}{4}\pi^2-4\right)X^2-\\
&-\frac{1}{3}(8x+22)Y^3X-\left(\left(\frac{27}{4}-\frac{11}{3}\pi^2\right)x-\frac{7}{3}\pi^2+\frac{57}{4}\right)YX+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(18-2\zeta_3+12x-\frac{5}{12}\pi^2\right)X+(6\zeta_3-12x-24)Y-\frac{19}{2}\zeta_3-\frac{23}{60}\pi^4+\\
&+\left(\frac{511}{64}-\frac{15}{4}\zeta_3+\frac{29}{48}\pi^2-\frac{49}{120}\pi^4\right)x+\frac{19}{8}\pi^2+\frac{511}{32}\bigg]\frac{x}{y}+\\
&+\bigg[6\mathrm{Li}_4\left(-\frac{x}{y}\right)+10\mathrm{Li}_4(-x)+2\mathrm{Li}_4(-y)-(1+3Y+3X)\mathrm{Li}_3(-x)-\\
&\quad-(8Y-6X)\mathrm{Li}_3(-y)+\left(X^2+(3Y+1)X+\frac{\pi^2}{3}\right)\mathrm{Li}_2(-x)+\frac{1}{24}X^4+\\
&\quad+\frac{7}{24}Y^4-\left(\frac{13}{24}-\frac{1}{6}Y\right)X^3+\left(\frac{9}{4}Y^2+\frac{7}{8}Y-\frac{7}{8}\pi^2+\frac{1}{8}\right)X^2-\\
&\quad-\left(\frac{11}{3}Y^3+\left(\frac{15}{2}-\frac{7}{6}\pi^2\right)Y+\zeta_3+\frac{\pi^2}{4}-6\right)X-\\
&\quad-(12-3\zeta_3)Y-\frac{35}{4}\zeta_3+\frac{85}{48}\pi^2-\frac{23}{120}\pi^4+\frac{511}{64}\bigg]\frac{1}{y}-\frac{\pi^2}{2}Y-\\
&-\frac{3y}{2x}Y^2-2xX^2\mathrm{Li}_2(-x)-\left[4Y^2\mathrm{Li}_2(-x)-\frac{9}{4}Y^2X+\frac{9}{4}Y^3-\frac{59}{8}Y^2\right]y+\\
&+i\pi\bigg\{\bigg[-(12+4x)\mathrm{Li}_3(-x)-(4+8x)\mathrm{Li}_3(-y)+\frac{1}{6}(5x+2)X^3-\\
&\qquad-\left(\frac{9}{2}x+4\right)X^2+((10+4x)X+4-(10+8x)Y)\mathrm{Li}_2(-x)-\\
&\qquad-\left(\left(\frac{11}{2}x+5\right)Y^2-(16+9x)Y+8x+\frac{25}{4}-\frac{\pi^2}{6}\right)X+\\
&\qquad+\left(\left(8+\frac{4}{3}\pi^2\right)x+\frac{5}{3}\pi^2+\frac{49}{4}\right)Y-6-\frac{7}{12}\pi^2+4\zeta_3\bigg]\frac{x}{y}+\\
&\quad+\bigg[-6\mathrm{Li}_3(-x)-2\mathrm{Li}_3(-y)+(1+5X-5Y)\mathrm{Li}_2(-x)+\frac{1}{6}X^3-\frac{5}{4}X^2+\\
&\qquad+\left(\frac{25}{4}Y+\frac{\pi^2}{12}-\frac{29}{4}-\frac{5}{2}Y^2\right)X+\left(\frac{5}{6}\pi^2+\frac{5}{4}\right)Y+2\zeta_3-6-\frac{\pi^2}{12}\bigg]\frac{1}{y}+\\
&\quad+\left(\frac{1}{2}YX^2+\frac{1}{6}Y^3-\frac{9}{2}Y^2\right)y-\frac{3}{xy}Y\bigg\},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
D_1^{[1]}=&\left[-\left(\frac{8}{9}+\frac{29}{36}x\right)X^2-\pi^2\left(\frac{11}{72}x+\frac{1}{12}\right)X\right]\frac{x}{y}-\frac{1}{9}xX^3+\\
&+\left[\frac{1}{18}X^3-\frac{19}{36}X^2-\frac{\pi^2}{24}X\right]\frac{1}{y}-\frac{1}{3}yX\mathrm{Li}_2(-x)-\frac{1}{27}(7+31x)X-\\
&-\frac{17}{72}\pi^2-\frac{25}{72}\pi^2x+\left[\frac{1}{3}\mathrm{Li}_3(-x)-\frac{1}{6}YX^2-\left(\frac{455}{108}+\frac{49}{36}\zeta_3\right)\right]y+\\
&+i\pi\bigg\{-\left[\frac{29}{18}x+\frac{16}{9}\right]\frac{x}{y}X-\frac{1}{3}xX^2+\left[\frac{1}{6}X^2-\frac{19}{18}X\right]\frac{1}{y}-\\
&\quad-\frac{1}{3}y\mathrm{Li}_2(-x)-\frac{31}{27}x-\frac{7}{27}-\left[\frac{1}{3}XY-\frac{5}{72}\pi^2\right]y\bigg\},
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
E_1^{[1]} = & \left[\left(\frac{16}{9}+\frac{29}{18}x\right)X^2+\left(\frac{11}{36}\pi^2 x+\frac{\pi^2}{6}\right)X\right]\frac{x}{y}+\frac{2}{9}xX^3- \\
& -\left[\frac{1}{9}X^3-\frac{19}{18}X^2-\frac{\pi^2}{12}X\right]\frac{1}{y}+\frac{2}{3}yX\mathrm{Li}_2(-x)+\frac{1}{27}(14+62x)X+ \\
& +\left[\frac{2}{3}(\mathrm{Li}_3(-y)-\mathrm{Li}_3(-x)+Y\mathrm{Li}_2(-x))+\frac{1}{3}YX^2+\frac{1}{3}Y^2X+\frac{2}{9}Y^3-\frac{29}{18}Y^2+\right. \\
& \quad \left.+\left(\frac{62}{27}-\frac{5}{12}\pi^2\right)Y-\left(\frac{685}{162}+\frac{35}{36}\zeta_3\right)\right]y-\frac{\pi^2}{8}+\frac{7}{72}\pi^2 x+ \\
& +i\pi\left\{\frac{1}{9}(32+29x)\frac{x}{y}X+\frac{2}{3}xX^2-\left[\frac{1}{3}X^2-\frac{19}{9}X\right]\frac{1}{y}+\frac{4}{3}y\mathrm{Li}_2(-x)-\right. \\
& \quad \left.-\frac{16}{9}+\left[\frac{2}{3}XY+\frac{2}{3}Y^2-\frac{29}{9}Y-\frac{\pi^2}{9}\right]y\right\},
\end{aligned}
\tag{A.5}
$$

$$
F_1^{[1]} = \frac{25}{81}y\,, \tag{A.6}
$$

For $h=1$ in eq. (2.11) and color factor $\mathrm{Tr}^{[2]}$ in eq. (2.19):

$$
\begin{aligned}
G_1^{[2]} = & \left[(2-4x)\mathrm{Li}_4\left(-\frac{x}{y}\right)+(2x+8)\mathrm{Li}_4(-x)-(12+x)\mathrm{Li}_4(-y)-\right. \\
& -\frac{5}{24}xY^4-\left((2-3x)Y+\frac{3}{2}x\right)\mathrm{Li}_3(-x)-\left(\frac{1}{6}-\frac{1}{12}x\right)YX^3- \\
& -\left((2+3x)X-(6+3x)Y-\frac{38}{3}-\frac{10}{3}x\right)\mathrm{Li}_3(-y)+\left(\frac{5}{6}+\frac{5}{4}x\right)Y^3X+ \\
& +\left(\frac{3}{2}xX+\frac{1}{3}(38+10x)Y+3\pi^2+\frac{5}{3}\pi^2 x\right)\mathrm{Li}_2(-x)- \\
& -\left(\left(\frac{5}{4}+\frac{13}{8}x\right)Y^2-\left(1+\frac{3}{4}x\right)Y+\left(\frac{1}{4}+\frac{\pi^2}{8}\right)x-1-\frac{\pi^2}{2}\right)X^2- \\
& -\left(\left(\frac{5}{8}-2\pi^2\right)x-\frac{7}{3}\pi^2+3\right)YX-\left(\left(\frac{\pi^2}{4}-3\zeta_3\right)x+\frac{23}{12}\pi^2-2\zeta_3\right)X- \\
& -\pi^2\left(\frac{3}{8}x-\frac{1}{12}\right)Y^2+\left(\left(\frac{79}{27}-\frac{5}{2}\zeta_3\right)x+3\zeta_3+\frac{158}{27}\right)Y-\frac{49}{144}\pi^4+\frac{95}{18}\pi^2+ \\
& \left.+\left(-\frac{131}{480}\pi^4-\frac{23213}{5184}+\frac{89}{36}\pi^2-\frac{197}{72}\zeta_3\right)x-\frac{485}{36}\zeta_3-\frac{23213}{2592}\right]\frac{x}{y}+ \\
& +\left[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)-3\mathrm{Li}_4(-y)+(2Y+1-2X)\mathrm{Li}_3(-x)+\right. \\
& \quad +\left(3Y+\frac{28}{3}-X\right)\mathrm{Li}_3(-y)+\left(\frac{1}{2}X^2-(1+Y)X+\pi^2+\frac{28}{3}Y\right)\mathrm{Li}_2(-x)- \\
& \quad -\frac{1}{12}YX^3+\left(-\frac{5}{8}Y^2+\frac{\pi^2}{4}+\frac{1}{8}Y\right)X^2-\frac{1}{8}Y^4-\frac{5}{24}\pi^2Y^2+ \\
& \quad +\left(\frac{79}{27}-\frac{3}{2}\zeta_3\right)Y+\left(\frac{11}{12}Y^3-\left(\frac{19}{8}-\frac{7}{6}\pi^2\right)Y-\pi^2+\zeta_3\right)X- \\
& \quad \left.-\frac{737}{72}\zeta_3+\frac{101}{36}\pi^2-\frac{161}{1440}\pi^4-\frac{23213}{5184}\right]\frac{1}{y}-
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{65}{12}+\frac{29}{12}x\right)Y^2X + 4xX\,\mathrm{Li}_3(-x) - x(X^2-2YX)\,\mathrm{Li}_2(-x) + \\
&+\pi^2\left(\frac{49}{16}+\frac{139}{48}x\right)Y + \frac{3y}{2x}Y^2 + \left[\frac{1}{2}Y^2\mathrm{Li}_2(-x)+\frac{11}{18}Y^3-\frac{277}{72}Y^2\right]y + \\
&+ i\pi\Bigg\{\Bigg[-(6+x)\,\mathrm{Li}_3(-x)+4\,\mathrm{Li}_3(-y)+\left(\frac{38}{3}+\frac{29}{6}x-xY\right)\mathrm{Li}_2(-x) - \\
&\qquad -\left(\frac{1}{6}-\frac{1}{12}x\right)X^3 + X^2 - \left(\left(\frac{599}{72}+\frac{\pi^2}{4}\right)x+\frac{277}{18}+\frac{5}{6}\pi^2\right)Y + \\
&\qquad +\left(\left(1-\frac{1}{2}x\right)Y^2+\frac{1}{2}Y+\left(\frac{7}{12}\pi^2-\frac{9}{8}\right)x+\pi^2-1\right)X + \\
&\qquad +\left(\frac{1}{2}\zeta_3-\frac{61}{144}\pi^2+\frac{79}{27}\right)x+5\zeta_3+\frac{158}{27}-\frac{31}{72}\pi^2\Bigg]\frac{x}{y} + \\
&\qquad +\Bigg[2\,\mathrm{Li}_3(-y)+\frac{25}{3}\mathrm{Li}_2(-x)-\frac{1}{12}X^3+\frac{5}{8}X^2-\left(\frac{293}{72}+\frac{5}{12}\pi^2\right)Y + \\
&\qquad +\left(\frac{\pi^2}{2}+\frac{1}{4}Y-\frac{19}{8}+\frac{1}{2}Y^2\right)X-\frac{1}{2}\zeta_3+\frac{79}{27}-\frac{13}{144}\pi^2\Bigg]\frac{1}{y} - \\
&\qquad -\left(3x+\frac{3}{2}\right)YX+\left[-\frac{1}{2}YX^2-\frac{1}{4}Y^3+\frac{31}{12}Y^2\right]y+\frac{3}{xy}Y\Bigg\}, \qquad\qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
H_1^{[2]} = &\Bigg[4x\,\mathrm{Li}_4\left(-\frac{x}{y}\right)-(3x+9)\,\mathrm{Li}_4(-x)-4x\,\mathrm{Li}_4(-y)+\left(\frac{1}{24}x-\frac{1}{4}\right)Y^4 + \\
&+\left((6+5x)X-(2+3x)Y-\frac{1}{3}x-\frac{13}{6}\right)\mathrm{Li}_3(-x) + \\
&+\left(-(2-3x)X+(4+2x)Y+\frac{1}{3}x+\frac{2}{3}\right)\mathrm{Li}_3(-y)+\left(\frac{5}{12}x+\frac{13}{6}\right)Y^3X + \\
&+\left(\left(2xY+\frac{x}{3}+\frac{13}{6}\right)X+\frac{1}{3}(2+x)Y-\pi^2(4+3x)\right)\mathrm{Li}_2(-x) - \\
&-\left(\left(\frac{1}{4}x-\frac{1}{6}\right)Y+\frac{11}{18}x+\frac{25}{36}\right)X^3+\left(\left(\frac{241}{27}+\frac{7}{2}\zeta_3\right)x+3\zeta_3+\frac{482}{27}\right)Y + \\
&+\left(\left(-\frac{1}{4}+\frac{15}{8}x\right)Y^2-\frac{1}{12}(7x+5)Y+\left(\frac{277}{72}+\frac{5}{24}\pi^2\right)x+\frac{37}{18}-\frac{\pi^2}{4}\right)X^2 + \\
&+\left(\left(-\frac{10}{3}\pi^2+\frac{23}{8}\right)x+\frac{31}{4}-\frac{10}{3}\pi^2\right)YX+\left(\frac{9}{8}\pi^2x+\frac{19}{12}\pi^2\right)Y^2 + \\
&+\left(\left(-\frac{79}{27}+\frac{409}{144}\pi^2-\frac{7}{2}\zeta_3\right)x+\frac{103}{24}\pi^2-3\zeta_3-\frac{62}{27}\right)X+\frac{551}{36}\zeta_3+\frac{19}{24}\pi^2 + \\
&+\left(-\frac{30659}{1296}+\frac{443}{72}\zeta_3+\frac{91}{144}\pi^2+\frac{179}{480}\pi^4\right)x+\frac{209}{720}\pi^4-\frac{30659}{648}\Bigg]\frac{x}{y} + \\
&+\Bigg[-\frac{9}{2}\mathrm{Li}_4(-x)-\left(Y+\frac{7}{3}-3X\right)\mathrm{Li}_3(-x)+\left(2Y+\frac{1}{3}-X\right)\mathrm{Li}_3(-y) + \\
&\quad+\left(\frac{1}{3}Y-\frac{3}{4}X^2+\frac{7}{3}X-2\pi^2\right)\mathrm{Li}_2(-x)+\frac{1}{48}X^4-\left(\frac{7}{18}-\frac{1}{12}Y\right)X^3 +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{125}{72}-\frac{1}{8}Y^2-\frac{\pi^2}{8}+\frac{5}{12}Y\right)X^2-\frac{1}{8}Y^4+\frac{19}{24}\pi^2Y^2+\left(\frac{241}{27}+\frac{3}{2}\zeta_3\right)Y+\\
&+\left(\frac{13}{12}Y^3-\left(\frac{5}{3}\pi^2-\frac{39}{8}\right)Y+\frac{17}{27}-\frac{3}{2}\zeta_3+\frac{95}{48}\pi^2\right)X+\frac{23}{144}\pi^2+\\
&+\frac{209}{1440}\pi^4+\frac{695}{72}\zeta_3-\frac{30659}{1296}\Bigg]\frac{1}{y}-\frac{1}{24}xX^4+\frac{3}{2}xX^2\mathrm{Li}_2(-x)+\\
&+\left[2Y^2\mathrm{Li}_2(-x)-\frac{4}{3}Y^2X+\frac{103}{36}Y^3-\frac{157}{18}Y^2-\frac{139}{48}\pi^2Y\right]y+2\frac{y}{x}Y^2+\\
&+i\pi\Bigg\{\Bigg[(4+2x)\mathrm{Li}_3(-x)+(5x+2)\mathrm{Li}_3(-y)+\left(\frac{1}{3}+\frac{1}{12}x\right)X^3+\\
&\quad+\left(-(x+3)X+(8+6x)Y+\frac{17}{6}+\frac{2}{3}x\right)\mathrm{Li}_2(-x)-\frac{1}{12}(43+31x)X^2+\\
&\quad+\left(\frac{7}{2}x+3\right)Y^2X-\left(\frac{25}{6}x+\frac{41}{6}\right)YX+\left(\frac{761}{72}-\frac{3}{4}\pi^2\right)xX+\\
&\quad+\left(\frac{427}{36}-\frac{7}{6}\pi^2\right)X-\left(\left(\frac{\pi^2}{4}+\frac{1049}{72}\right)x-\frac{\pi^2}{6}+\frac{833}{36}\right)Y+\\
&\quad+\left(6-\frac{\pi^2}{18}\right)x+\frac{140}{9}-\frac{4}{9}\pi^2\Bigg]\frac{x}{y}+\\
&\quad+\Bigg[2\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+\left(\frac{8}{3}-\frac{3}{2}X+4Y\right)\mathrm{Li}_2(-x)+\frac{1}{6}X^3-\frac{23}{12}X^2+\\
&\qquad+\left(-\frac{13}{6}Y-\frac{7}{12}\pi^2+\frac{601}{72}+\frac{3}{2}Y^2\right)X+\left(\frac{\pi^2}{12}-\frac{329}{72}\right)Y+\\
&\qquad+\frac{86}{9}-\frac{17}{36}\pi^2\Bigg]\frac{1}{y}\left[\frac{1}{2}YX^2-\frac{5}{12}Y^3+\frac{85}{12}Y^2\right]y+\frac{4}{xy}Y\Bigg\}, \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
I_1^{[2]}=&\Bigg[-4\mathrm{Li}_4\left(-\frac{x}{y}\right)-(7+x)\mathrm{Li}_4(-x)-(3x+2)\mathrm{Li}_4(-y)-(4X-(6+3x)Y)\mathrm{Li}_3(-y)+\\
&+\left((2+x)X+\frac{3}{2}+2Y\right)\mathrm{Li}_3(-x)-\left(\frac{1}{24}+\frac{1}{12}x\right)X^4+\left(\frac{3}{4}x+\frac{7}{12}-\frac{1}{6}xY\right)X^3-\\
&-\frac{1}{6}Y^4+Y^3X-\left(\left(2Y+\frac{3}{2}\right)X+\frac{2}{3}\pi^2+\frac{2}{3}\pi^2x\right)\mathrm{Li}_2(-x)+\left(\frac{\pi^2}{12}x-\frac{\pi^2}{6}\right)Y^2+\\
&+\left(-\left(\frac{3}{2}-\frac{1}{4}x\right)Y^2-\frac{1}{4}(3x+5)Y+\left(\frac{2}{3}\pi^2+\frac{7}{8}\right)x-\frac{1}{2}+\frac{7}{12}\pi^2\right)X^2-\\
&-\left(\left(\frac{4}{3}\pi^2-\frac{9}{4}\right)x-\frac{19}{4}+\pi^2\right)YX-\left(9+6x-\frac{\pi^2}{4}\right)X+\frac{17}{2}\zeta_3+\frac{13}{60}\pi^4+\\
&+(12+6x-2\zeta_3)Y+\left(-\frac{511}{64}+\frac{7}{40}\pi^4+\frac{15}{4}\zeta_3-\frac{29}{48}\pi^2\right)x-\frac{511}{32}-\frac{41}{24}\pi^2\Bigg]\frac{x}{y}+\\
&+\Bigg[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-\frac{7}{2}\mathrm{Li}_4(-x)-\mathrm{Li}_4(-y)-\frac{1}{48}X^4-\frac{1}{12}Y^4-(2X-3Y)\mathrm{Li}_3(-y)+\\
&\quad+\left(\frac{1}{2}+Y+X\right)\mathrm{Li}_3(-x)-\left(\frac{1}{4}X^2+\left(\frac{1}{2}+Y\right)X+\frac{\pi^2}{3}\right)\mathrm{Li}_2(-x)+\frac{5}{24}X^3+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{7}{24}\pi^2-\frac{3}{4}Y^2-\frac{3}{8}Y+\frac{5}{8}\right)X^2+\left(\frac{2}{3}Y^3+\left(\frac{5}{2}-\frac{\pi^2}{2}\right)Y-3+\frac{\pi^2}{12}\right)X-\\
&-\frac{\pi^2}{12}Y^2+(6-\zeta_3)Y-\frac{53}{48}\pi^2+\frac{23}{4}\zeta_3+\frac{13}{120}\pi^4-\frac{511}{64}\bigg]\frac{1}{y}+\frac{1}{2}xX^2\mathrm{Li}_2(-x)-\\
&-\left(\frac{2}{3}+x\right)Y^3X+\frac{\pi^2}{6}Y+\left[\frac{3}{2}Y^2\mathrm{Li}_2(-x)-\frac{3}{4}Y^2X+\frac{3}{4}Y^3+\left(\frac{1}{2x}-\frac{25}{8}\right)Y^2\right]y+\\
&+i\pi\Bigg\{\bigg[(4+x)\mathrm{Li}_3(-x)+(3x+2)\mathrm{Li}_3(-y)+\left(\frac{3}{2}x+\frac{5}{4}\right)X^2-\left(\frac{1}{3}x+\frac{1}{6}\right)X^3-\\
&\qquad-\left((x+3)X+\frac{3}{2}-(3x+4)Y\right)\mathrm{Li}_2(-x)-\left(3x+\frac{11}{2}\right)YX-2\zeta_3+\\
&\qquad+\left(\frac{15}{4}-\frac{\pi^2}{6}+4x\right)X-\left(\left(4+\frac{\pi^2}{2}\right)x+\frac{27}{4}+\frac{2}{3}\pi^2\right)Y+3+\frac{\pi^2}{4}\bigg]\frac{x}{y}+\\
&\qquad+\bigg[2\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)-\left(\frac{3}{2}X+\frac{1}{2}-2Y\right)\mathrm{Li}_2(-x)-\frac{1}{12}X^3+\frac{1}{2}X^2+\\
&\qquad\quad+\left(\frac{15}{4}-\frac{9}{4}Y-\frac{\pi^2}{12}+Y^2\right)X-\left(\frac{7}{4}+\frac{\pi^2}{3}\right)Y-\zeta_3+3+\frac{\pi^2}{12}\bigg]\frac{1}{y}-\\
&\qquad-2xY^2X+\left(\frac{3}{2}xY^2+\frac{1}{y^2}Y\right)\frac{y}{x}\Bigg\}, \qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
J_1^{[2]}=&\bigg[-\frac{1}{3}\mathrm{Li}_3(-y)-\frac{1}{3}Y\mathrm{Li}_2(-x)-\frac{1}{6}Y^2X-\frac{1}{9}Y^3+\frac{29}{36}Y^2+\\
&+\left(\frac{5}{24}\pi^2-\frac{31}{27}\right)Y+\frac{49}{36}\zeta_3-\frac{25}{72}\pi^2\bigg]y-\frac{455}{108}(1+x)+\\
&+i\pi\left\{-\frac{1}{3}\mathrm{Li}_2(-x)-\frac{1}{3}Y^2+\frac{29}{18}Y-\left(\frac{\pi^2}{72}+\frac{31}{27}\right)\right\}y\,, \qquad \text{(A.10)}
\end{aligned}
$$

$$
\begin{aligned}
K_1^{[2]}=&\left[-\left(\frac{8}{9}+\frac{29}{36}x\right)X^2-\pi^2\left(\frac{11}{72}x+\frac{1}{12}\right)X\right]\frac{x}{y}-\frac{1}{9}xX^3-\frac{1}{3}yX\mathrm{Li}_2(-x)+\\
&+\left[\frac{1}{18}X^3-\frac{19}{36}X^2-\frac{\pi^2}{24}X\right]\frac{1}{y}-\frac{1}{27}(7+31x)X+\frac{\pi^2}{72}(1-7x)+\\
&+\bigg[\frac{1}{3}\mathrm{Li}_3(-x)-\frac{1}{3}\mathrm{Li}_3(-y)-\frac{1}{3}Y\mathrm{Li}_2(-x)-\frac{1}{6}YX^2-\frac{1}{6}Y^2X-\frac{1}{9}Y^3+\\
&\quad+\frac{29}{36}Y^2+\left(\frac{5}{24}\pi^2-\frac{31}{27}\right)Y+\left(\frac{685}{162}+\frac{35}{36}\zeta_3\right)\bigg]y+\\
&+i\pi\Bigg\{-\left(\frac{16}{9}+\frac{29}{18}x\right)\frac{x}{y}X-\frac{1}{3}xX^2+\frac{8}{9}-\left[\frac{19}{18}X-\frac{1}{6}X^2\right]\frac{1}{y}-\\
&\qquad-\frac{2}{3}y\mathrm{Li}_2(-x)+\left[-\frac{1}{3}Y^2+\frac{29}{18}Y-\frac{1}{3}XY+\frac{\pi^2}{18}\right]y\Bigg\}, \qquad \text{(A.11)}
\end{aligned}
$$

$$
L_1^{[2]}=-\frac{25}{81}y\,, \qquad \text{(A.12)}
$$

For $h=2$ in eq. (2.12) and color factor $\mathrm{Tr}^{[1]}$ in eq. (2.19):

$$
\begin{aligned}
A_2^{[1]} = & -\left(\frac{56}{9}+\frac{85}{18}x\right)\frac{x}{y}X^2 - 3\mathrm{Li}_4\left(-\frac{x}{y}\right) - (3-x)\mathrm{Li}_4(-x) + \frac{1}{2}xX^2\mathrm{Li}_2(-x) + \\
& + 3\mathrm{Li}_4(-y) - \left(\frac{11}{6}x - 3 + xX\right)\mathrm{Li}_3(-x) - 3X\mathrm{Li}_3(-y) + \left(-\frac{49}{36}x + \frac{1}{6}Yx\right)X^3 + \\
& + \left(\left(\frac{11}{6}x-3\right)X - \frac{\pi^2}{2} - \frac{\pi^2}{3}x\right)\mathrm{Li}_2(-x) + \frac{1}{12}xX^4 - \frac{\pi^2}{3}xYX + \\
& + \left(\left(\frac{11}{12}x - \frac{3}{2}\right)Y - \frac{5}{12}\pi^2 x\right)X^2 + \left(-\left(\frac{1}{2}\zeta_3 - \frac{373}{144}\pi^2 + \frac{79}{27}\right)x + 3\zeta_3 + \frac{\pi^2}{2}\right)X + \\
& + \left(\frac{23213}{5184} + \frac{113}{1440}\pi^4 - \frac{49}{18}\pi^2 + \frac{197}{72}\zeta_3\right)x - \frac{\pi^4}{30} - 3\zeta_3 - \frac{\pi^2}{2} + \\
& + \Bigg[-3\mathrm{Li}_4(-x) + 3X\zeta_3 + 3\mathrm{Li}_4(-y) - 3\mathrm{Li}_4\left(-\frac{x}{y}\right) - \frac{\pi^4}{30} - \\
& \quad - \frac{\pi^2}{2}\mathrm{Li}_2(-x) - 3X\mathrm{Li}_3(-y)\Bigg]\frac{1}{x} + \left[\frac{3}{4}Y^2X^2 + \frac{1}{8}Y^4 + \frac{\pi^2}{4}Y^2 - \frac{1}{2}XY^3\right]\frac{y}{x} + \\
& + i\pi\Bigg\{-\left(\frac{85}{9}x + \frac{112}{9}\right)\frac{x}{y}X - x\mathrm{Li}_3(-x) - 3\mathrm{Li}_3(-y) + xX\mathrm{Li}_2(-x) + \\
& \qquad + \left(\frac{11}{6}x - 3\right)\mathrm{Li}_2(-x) + \frac{1}{3}xX^3 - \left(\frac{49}{12} - \frac{1}{2}Y\right)xX^2 + \left(\frac{11}{6}x - 3\right)YX - \\
& \qquad - \frac{\pi^2}{6}xX - \left(\frac{19}{144}\pi^2 + \frac{79}{27} + \frac{1}{2}\zeta_3\right)x + \frac{\pi^2}{2} + 3\zeta_3 + \\
& \qquad + \left(3\zeta_3 - 3\mathrm{Li}_3(-y)\right)\frac{1}{x}\Bigg\}, \qquad\qquad (A.13)
\end{aligned}
$$

$$
\begin{aligned}
B_2^{[1]} = & \left(\frac{869}{72} + \frac{617}{72}x\right)\frac{x}{y}X^2 - \frac{3}{2}xX^2\mathrm{Li}_2(-x) + 4\zeta_3 - \frac{8}{9}\pi^2 - \frac{\pi^4}{45} + 2\mathrm{Li}_4(-y) + \\
& + 2\mathrm{Li}_4\left(-\frac{x}{y}\right) - \frac{1}{4}xX^4 - (3x-2)\mathrm{Li}_4(-x) - \left(2Y + 2 + \frac{11}{3}x - 4X\right)\mathrm{Li}_3(-y) + \\
& + \left(-6 + \frac{11}{3}x + 4Y + 3xX\right)\mathrm{Li}_3(-x) + \left(-\frac{1}{2}Yx + \frac{125}{36}x\right)X^3 + \\
& + \left(\left(6 - \frac{11}{3}x - 2Y\right)X - \left(2 + \frac{11}{3}x\right)Y + \pi^2 x + \frac{5}{3}\pi^2\right)\mathrm{Li}_2(-x) + \\
& + \left(\left(\frac{1}{2} + \frac{1}{4}x\right)Y^2 + \left(3 - \frac{31}{12}x\right)Y + \frac{3}{2}\pi^2 x\right)X^2 - \frac{2}{3}Y^3X - \left(\frac{31}{12}x + 2\right)Y^2X + \\
& + \left(\left(-\frac{\pi^2}{3} + \frac{599}{72}\right)x - \frac{5}{12}\pi^2 + \frac{271}{36}\right)Y^2 - \left(\frac{1}{4} - \left(\frac{9}{4} - \frac{2}{3}\pi^2\right)x\right)YX + \\
& + \left(\left(-\frac{373}{72}\pi^2 - \frac{4}{27} + \zeta_3\right)x - \frac{5}{6}\pi^2 - 4\zeta_3\right)X + \left(\frac{1}{4} + \frac{1}{8}x\right)Y^4 - \\
& - \left(\frac{17}{9} + \frac{71}{36}x\right)Y^3 + \left(\left(\frac{139}{24}\pi^2 - \frac{158}{27} - \zeta_3\right)x - \frac{64}{9} + \frac{211}{36}\pi^2 + 2\zeta_3\right)Y + \\
& + \left(-\frac{55}{144}\pi^2 + \frac{30659}{1296} + \frac{79}{1440}\pi^4 - \frac{443}{72}\zeta_3\right)x +
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[4\mathrm{Li}_4\left(-\frac{x}{y}\right)+4\mathrm{Li}_4(-x)-2\mathrm{Li}_4(-y)+2Y\mathrm{Li}_3(-x)-(Y+1-5X)\mathrm{Li}_3(-y)-\\
&\quad-\left(Y+YX-\frac{4}{3}\pi^2\right)\mathrm{Li}_2(-x)+Y^2X^2-\left(5\zeta_3+\frac{7}{8}Y^2+\frac{5}{6}Y^3\right)X+\frac{1}{4}Y^4-\\
&\quad-\frac{59}{72}Y^3+\left(\frac{73}{18}+\frac{\pi^2}{24}\right)Y^2+\left(\zeta_3+\frac{43}{18}\pi^2\right)Y+\frac{\pi^4}{45}+\zeta_3\Bigg]\frac{1}{x}+\\
&+i\pi\Bigg\{\left[(3x+5)X\mathrm{Li}_2(-x)+\left(\frac{797}{36}+\frac{134}{9}x\right)X\right]\frac{x}{y}+\left[\frac{1}{4}X+3X\mathrm{Li}_2(-x)\right]\frac{1}{y}-\\
&\qquad-\frac{64}{9}+(3x+4)\mathrm{Li}_3(-x)+2\mathrm{Li}_3(-y)-\left(\frac{22}{3}x-4+2Y\right)\mathrm{Li}_2(-x)-\\
&\qquad-xX^3+\left(\frac{29}{3}-Y\right)xX^2+\left(\frac{2}{3}+\frac{1}{2}x\right)Y^3+\frac{20}{3}yY^2+\\
&\qquad+\left(\left(-2+\frac{1}{2}x\right)Y^2+\left(-\frac{20}{3}x+4\right)Y+\frac{\pi^2}{3}x\right)X+\\
&\qquad+\left(\left(\frac{170}{9}-\frac{\pi^2}{3}\right)x+\frac{533}{36}+\frac{\pi^2}{6}\right)Y+\left(\frac{11}{18}\pi^2-6\right)x-2\zeta_3-\frac{3}{4}\pi^2+\\
&\qquad+\Bigg[2\mathrm{Li}_3(-x)+4\mathrm{Li}_3(-y)-(Y+1)\mathrm{Li}_2(-x)-\left(Y^2+\frac{3}{4}Y\right)X+\frac{1}{3}Y^3-\\
&\qquad\quad-\frac{17}{6}Y^2+\left(\frac{\pi^2}{12}+\frac{73}{9}\right)Y-4\zeta_3\Bigg]\frac{1}{x}+\frac{1}{xy}X\mathrm{Li}_2(-x)\Bigg\}, \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
C_2^{[1]} =\ &\frac{x}{8y}(5x-7)X^2+12\mathrm{Li}_4\left(-\frac{x}{y}\right)+(4x+12)\mathrm{Li}_4(-x)+2xX^2\mathrm{Li}_2(-x)+\\
&+8x\mathrm{Li}_4(-y)-(4xX+12Y)\mathrm{Li}_3(-x)+(6X+2-(8x+10)Y)\mathrm{Li}_3(-y)+\\
&+\left(6YX-4(1+x)Y^2+2Y+\frac{4}{3}\pi^2x-2\pi^2\right)\mathrm{Li}_2(-x)+\frac{5}{24}xX^4+\\
&+\left(\frac{2}{3}Yx-\frac{9}{4}x\right)X^3+\left(\left(-\frac{3}{4}x+3\right)Y^2+\frac{9}{4}xY-\frac{11}{6}\pi^2x\right)X^2-\\
&-\left(\frac{8}{3}x+5\right)Y^3X+\left(4+\frac{9}{4}x\right)Y^2X+\left(\frac{3}{4}+\left(\frac{11}{3}\pi^2-\frac{27}{4}\right)x\right)YX+\\
&+\left(-\frac{\pi^2}{2}+12x-6\zeta_3\right)X+\left(\frac{1}{24}x+\frac{2}{3}\right)Y^4-\left(\frac{9}{4}x+\frac{7}{3}\right)Y^3+\\
&+\left(\left(\frac{59}{8}-\frac{\pi^2}{2}\right)x+\frac{3}{4}+\frac{13}{12}\pi^2\right)Y^2+\left(-\frac{5}{4}\pi^2-6+10\zeta_3-12x\right)Y+\\
&+\left(\frac{511}{64}+\frac{29}{48}\pi^2-\frac{49}{120}\pi^4-\frac{15}{4}\zeta_3\right)x+\frac{\pi^2}{6}-2\zeta_3+\\
&+\Bigg[6\mathrm{Li}_4\left(-\frac{x}{y}\right)+6\mathrm{Li}_4(-x)-6Y\mathrm{Li}_3(-x)+(3-5Y+3X)\mathrm{Li}_3(-y)+\frac{3}{2}Y^2X^2+\\
&\quad+(3YX-\pi^2+3Y-2Y^2)\mathrm{Li}_2(-x)+\left(\frac{21}{8}Y^2-\frac{5}{2}Y^3-3\zeta_3\right)X+\frac{1}{3}Y^4-\\
&\quad-\frac{7}{24}Y^3+\left(\frac{13}{24}\pi^2+\frac{3}{2}\right)Y^2+\left(-\frac{5}{3}\pi^2+5\zeta_3\right)Y-3\zeta_3\Bigg]\frac{1}{x}+
\end{aligned}
$$

$$
\begin{aligned}
&+i\pi\Bigg\{\Bigg[-(4x+10)X\mathrm{Li}_2(-x)+\left(\frac{17}{4}+8x\right)X\Bigg]\frac{x}{y}-\left[\frac{3}{4}X+9X\mathrm{Li}_2(-x)\right]\frac{1}{y}-\\
&\quad-(12+4x)\mathrm{Li}_3(-x)-(8x+4)\mathrm{Li}_3(-y)+(2-(8x+2)Y)\mathrm{Li}_2(-x)+\\
&\quad+\frac{5}{6}xX^3+\frac{1}{2}(Y-9)xX^2-\left(\left(\frac{11}{2}x+2\right)Y^2-(6+9x)Y\right)X+\\
&\quad+\left(\frac{2}{3}+\frac{1}{6}x\right)Y^3-\left(\frac{9}{2}x+4\right)Y^2+\left(\left(8+\frac{4}{3}\pi^2\right)x+\frac{9}{4}+\frac{3}{2}\pi^2\right)Y+4\zeta_3+\\
&\quad+\Bigg[-6\mathrm{Li}_3(-x)-2\mathrm{Li}_3(-y)-(Y-3)\mathrm{Li}_2(-x)-\left(Y^2-\frac{9}{4}Y\right)X+\frac{1}{3}Y^3+\\
&\qquad+\frac{1}{4}Y^2+\left(\frac{3}{4}\pi^2+3\right)Y+2\zeta_3\Bigg]\frac{1}{x}-\frac{3}{xy}X\mathrm{Li}_2(-x)-\frac{5}{12}\pi^2-6\Bigg\}, \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
D_2^{[1]} =& \Bigg[\frac{1}{3}(\mathrm{Li}_3(-x)-X\mathrm{Li}_2(-x))+\frac{1}{9}X^3-\left(\frac{29}{36}+\frac{1}{6}Y\right)X^2+\frac{25}{72}\pi^2-\frac{455}{108}-\frac{49}{36}\zeta_3-\\
&-\left(\frac{11}{72}\pi^2-\frac{31}{27}\right)X\Bigg]x+\\
&+i\frac{\pi}{3}x\left\{X^2-\mathrm{Li}_2(-x)-\left(\frac{29}{6}+Y\right)X+\frac{31}{9}+\frac{5}{24}\pi^2\right\}, \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
E_2^{[1]} =& \frac{2}{3}x(\mathrm{Li}_3(-y)-\mathrm{Li}_3(-x)+(X+Y)\mathrm{Li}_2(-x))-\frac{2}{9}xX^3+\left(\frac{1}{3}Y+\frac{29}{18}\right)xX^2+\\
&+\left(\frac{1}{3}Y^2-\frac{62}{27}+\frac{11}{36}\pi^2\right)xX-\frac{2}{9}yY^3-\left(\frac{13}{9}+\frac{29}{18}x\right)Y^2+\\
&+\left(\left(-\frac{5}{12}\pi^2+\frac{62}{27}\right)x+\frac{16}{9}-\frac{4}{9}\pi^2\right)Y+\left(-\frac{685}{162}-\frac{35}{36}\zeta_3-\frac{7}{72}\pi^2\right)x+\\
&+\frac{2}{9}\pi^2+\frac{1}{9}\left(Y^3-8Y^2-2\pi^2Y\right)\frac{1}{x}+\\
&+i\pi\left\{\frac{16}{9}+\frac{4}{3}x\mathrm{Li}_2(-x)-\frac{2}{3}xX^2+\left(\frac{29}{9}+\frac{2}{3}Y\right)xX-\frac{2}{3}yY^2-\right.\\
&\qquad\left.-\frac{1}{9}(26+29x)Y-\frac{\pi^2}{9}x-\left(\frac{16}{9}Y-\frac{1}{3}Y^2\right)\frac{1}{x}\right\}, \qquad \text{(A.17)}
\end{aligned}
$$

$$
F_2^{[1]} = \frac{25}{81}x\,, \qquad \text{(A.18)}
$$

For $h=2$ in eq. (2.12) and color factor $\mathrm{Tr}^{[2]}$ in eq. (2.19):

$$
\begin{aligned}
G_2^{[2]} =& -\frac{x}{4y}(3+x)X^2+xX^2\mathrm{Li}_2(-x)-(3+4x)\mathrm{Li}_4\left(-\frac{x}{y}\right)+\frac{1}{12}xX^3Y+\\
&+(2x-3)\mathrm{Li}_4(-x)-(4+x)\mathrm{Li}_4(-y)+\left(3-\frac{3}{2}x-4xX+3xY\right)\mathrm{Li}_3(-x)+\\
&+\left(3yX+(3x+4)Y+\frac{9}{2}+\frac{10}{3}x\right)\mathrm{Li}_3(-y)+\left(1+\frac{5}{4}x\right)Y^3X+\\
&+\left(\left(\frac{3}{2}x-3-2Yx\right)X-\frac{1}{2}yY^2+\left(\frac{9}{2}+\frac{10}{3}x\right)Y+\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\pi^2}{6}(10x+1)\Big)\mathrm{Li}_2(-x)+\bigg(\Big(-\frac{13}{8}+2\pi^2\Big)x+\frac{2}{3}\pi^2-\frac{3}{2}\bigg)YX-\\
&-\bigg(\Big(\frac{13}{8}x+\frac{3}{4}\Big)Y^2+\Big(\frac{3}{2}-\frac{3}{4}x\Big)Y+\frac{\pi^2}{8}x\bigg)X^2+\Big(\frac{17}{4}+\frac{29}{12}x\Big)Y^2X+\\
&+\bigg(\Big(-\frac{\pi^2}{4}+3\zeta_3\Big)x+\frac{\pi^2}{6}+3\zeta_3\bigg)X-\Big(\frac{1}{6}+\frac{5}{24}x\Big)Y^4+\Big(\frac{19}{36}+\frac{11}{18}x\Big)Y^3-\\
&-\bigg(\Big(\frac{241}{72}+\frac{3}{8}\pi^2\Big)x+\frac{95}{36}-\frac{\pi^2}{2}\bigg)Y^2-\Big(\frac{131}{480}\pi^4+\frac{23213}{5184}-\frac{107}{36}\pi^2+\frac{197}{72}\zeta_3\Big)x+\\
&+\bigg(\Big(\frac{79}{27}-\frac{139}{48}\pi^2-\frac{5}{2}\zeta_3\Big)x-\frac{179}{36}\pi^2-4\zeta_3+\frac{32}{9}\bigg)Y+\frac{61}{36}\pi^2+\frac{2}{45}\pi^4-4\zeta_3+\\
&+\bigg[-3\mathrm{Li}_4\Big(-\frac{x}{y}\Big)-3\mathrm{Li}_4(-x)-\frac{1}{2}\mathrm{Li}_4(-y)-\Big(3X-\frac{3}{2}-2Y\Big)\mathrm{Li}_3(-y)+\\
&\quad+\Big(\frac{1}{4}Y^2+\frac{3}{2}Y-\frac{\pi^2}{6}\Big)\mathrm{Li}_2(-x)-\frac{3}{4}Y^2X^2-\frac{7}{48}Y^4+\frac{11}{36}Y^3-\frac{3}{2}\zeta_3+\frac{\pi^4}{180}+\\
&\quad+\Big(3\zeta_3+\frac{3}{4}Y^3+\frac{\pi^2}{3}Y+\frac{3}{2}Y^2\Big)X-\Big(\frac{37}{36}-\frac{\pi^2}{8}\Big)Y^2-\Big(\frac{19}{9}\pi^2+2\zeta_3\Big)Y\bigg]\frac{1}{x}+\\
&+i\pi\bigg\{\frac{x}{8y}(13+9x)X+\frac{3}{2y}X-x\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+\frac{1}{12}xX^3-\frac{1}{2}xX^2Y+\\
&\quad+\Big((1-x)Y+\frac{3}{2}+\frac{29}{6}x\Big)\mathrm{Li}_2(-x)+\Big(1-\frac{1}{2}x\Big)Y^2X+(3x+1)YX+\\
&\quad+\frac{7}{12}\pi^2xX+\frac{1}{12}(31x+43)Y^2-\bigg(\Big(\frac{599}{72}+\frac{\pi^2}{4}\Big)x+\frac{\pi^2}{6}+\frac{61}{9}\bigg)Y+\\
&\quad+\Big(\frac{1}{2}\zeta_3+\frac{79}{27}-\frac{61}{144}\pi^2\Big)x-\zeta_3+\frac{\pi^2}{4}+\frac{32}{9}+\\
&\quad+\bigg[-\mathrm{Li}_3(-y)+\frac{1}{2}(Y+3)(\mathrm{Li}_2(-x)+YX)+\frac{5}{3}Y^2-\\
&\qquad-\Big(\frac{\pi^2}{12}+\frac{37}{18}\Big)Y+\zeta_3\bigg]\frac{1}{x}-\frac{y^2}{4x}Y^3\bigg\},
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
H_2^{[2]} = &-\frac{1}{72}\Big(277+241x\Big)\frac{x}{y}X^2-\frac{3}{2}xX^2\mathrm{Li}_2(-x)+4(x-1)\mathrm{Li}_4\Big(-\frac{x}{y}\Big)-\\
&-(3x+4)\mathrm{Li}_4(-x)-4x\mathrm{Li}_4(-y)+\Big(5xX+(4-3x)Y-\frac{1}{3}x\Big)\mathrm{Li}_3(-x)+\\
&+\Big((3x-2)X+(2x+4)Y+\frac{1}{3}x-4\Big)\mathrm{Li}_3(-y)+\frac{1}{24}xX^4-\Big(\frac{1}{4}Y+\frac{11}{18}\Big)xX^3+\\
&+\bigg(\Big(2(x-1)Y+\frac{1}{3}x\Big)X-2yY^2+\Big(\frac{1}{3}x-4\Big)Y-3\pi^2x\bigg)\mathrm{Li}_2(-x)+\\
&+\bigg(\Big(-1+\frac{15}{8}x\Big)Y^2-\frac{7}{12}xY+\frac{5}{24}\pi^2x\bigg)X^2+\Big(\frac{7}{3}+\frac{5}{12}x\Big)Y^3X-\\
&-\Big(\frac{4}{3}x+5\Big)Y^2X+\bigg(\Big(\frac{31}{8}-\frac{10}{3}\pi^2\Big)x-\frac{2}{3}\pi^2+\frac{5}{4}\bigg)YX-\Big(\frac{5}{12}-\frac{1}{24}x\Big)Y^4+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\left(\frac{409}{144}\pi^2-\frac{7}{2}\zeta_3-\frac{79}{27}\right)x+\frac{\pi^2}{2}+2\zeta_3\right)X+\left(\frac{103}{36}x+\frac{53}{18}\right)Y^3+\\
&+\left(\left(\frac{9}{8}\pi^2-\frac{83}{9}\right)x-\frac{5}{12}\pi^2-\frac{44}{9}\right)Y^2+\left(\frac{19}{144}\pi^2+\frac{443}{72}\zeta_3-\frac{30659}{1296}+\frac{179}{480}\pi^4\right)x+\\
&+\left(\left(\frac{7}{2}\zeta_3+\frac{241}{27}-\frac{139}{48}\pi^2\right)x+\frac{59}{9}-\frac{5}{36}\pi^2-4\zeta_3\right)Y-\frac{11}{36}\pi^2+4\zeta_3+\\
&+\Bigg[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)+2Y\mathrm{Li}_3(-x)-\left(\frac{5}{2}+X-2Y\right)\mathrm{Li}_3(-y)+\\
&\quad+\left(Y^2-\frac{5}{2}Y-YX\right)\mathrm{Li}_2(-x)-\frac{1}{2}Y^2X^2-\frac{5}{24}Y^4+\frac{55}{72}Y^3+\\
&\quad+\left(-\frac{\pi^2}{3}Y+\frac{7}{6}Y^3+\zeta_3-\frac{19}{8}Y^2\right)X-\left(\frac{127}{36}+\frac{5}{24}\pi^2\right)Y^2+\\
&\quad+\left(\frac{13}{18}\pi^2-2\zeta_3\right)Y+\frac{5}{2}\zeta_3\Bigg]\frac{1}{x}+\\
&+i\pi\Bigg\{\left[(x+3)X\mathrm{Li}_2(-x)-\left(\frac{923}{72}+\frac{761}{72}x\right)X\right]\frac{x}{y}-X\left[\frac{5}{4}-3\mathrm{Li}_2(-x)\right]\frac{1}{y}+\\
&\qquad+(2x+4)\mathrm{Li}_3(-x)+(5x+2)\mathrm{Li}_3(-y)+\left((2+6x)Y+\frac{2}{3}x-4\right)\mathrm{Li}_2(-x)+\\
&\qquad+\frac{1}{12}xX^3-\left(\frac{31}{12}-\frac{1}{2}Y\right)xX^2+\left(2+\frac{7}{2}x\right)Y^2X-\left(\frac{25}{6}x+6\right)YX-\\
&\qquad-\frac{3}{4}\pi^2xX-\left(\frac{5}{12}x+\frac{2}{3}\right)Y^3+\left(\frac{85}{12}x+\frac{35}{6}\right)Y^2-2\zeta_3-\\
&\qquad-\left(\left(\frac{\pi^2}{4}+\frac{1049}{72}\right)x+\frac{5}{6}\pi^2+\frac{307}{36}\right)Y+\left(-\frac{\pi^2}{18}+6\right)x+\frac{59}{9}+\frac{\pi^2}{4}+\\
&\qquad+\Bigg[2\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+\left(Y-\frac{5}{2}\right)\mathrm{Li}_2(-x)+\left(Y^2-\frac{9}{4}Y\right)X-\frac{1}{3}Y^3+\\
&\qquad\quad+\frac{7}{6}Y^2-\left(\frac{5}{12}\pi^2+\frac{127}{18}\right)Y-\zeta_3\Bigg]\frac{1}{x}+\frac{1}{xy}X\mathrm{Li}_2(-x)\Bigg\},
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
I_2^{[2]}=&\Bigg[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)-\frac{1}{2}\mathrm{Li}_4(-y)+2Y\mathrm{Li}_3(-x)+(2Y-X-1)\mathrm{Li}_3(-y)+\\
&\quad+\left(\frac{3}{4}Y^2+\frac{\pi^2}{3}-YX-Y\right)\mathrm{Li}_2(-x)-\frac{1}{2}Y^2X^2-\left(\frac{7}{8}Y^2-\zeta_3-\frac{11}{12}Y^3\right)X-\\
&\quad-\frac{5}{48}Y^4+\frac{1}{24}Y^3-\left(\frac{\pi^2}{4}+1\right)Y^2+\left(\frac{2}{3}\pi^2-2\zeta_3\right)Y+\frac{\pi^4}{180}+\zeta_3\Bigg]\frac{1}{x}-\\
&-4\mathrm{Li}_4\left(-\frac{x}{y}\right)-(4+x)\mathrm{Li}_4(-x)-(3x+1)\mathrm{Li}_4(-y)+(xX+4Y)\mathrm{Li}_3(-x)-\\
&-\left(2X+\frac{1}{2}-(3x+4)Y\right)\mathrm{Li}_3(-y)-\frac{1}{2}xX^2\mathrm{Li}_2(-x)-\frac{x}{8y}(7x+3)X^2+\zeta_3-\frac{\pi^2}{6}+\\
&+\frac{\pi^4}{90}+\left(3-4\zeta_3+\frac{\pi^2}{2}+6x\right)Y+\left(\left(\frac{\pi^2}{12}-\frac{25}{8}\right)x-\frac{\pi^2}{2}-\frac{3}{4}\right)Y^2+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{4}x-1\right)Y^2X^2+\left(-\frac{3}{4}xY+\frac{2}{3}\pi^2x\right)X^2+\left(\frac{11}{6}+x\right)Y^3X-\frac{1}{4}(3x+5)Y^2X-\\
&-\left(\frac{1}{4}+\left(\frac{4}{3}\pi^2-\frac{9}{4}\right)x\right)YX-\left(6x-2\zeta_3-\frac{\pi^2}{6}\right)X-\frac{3}{4}yY^3+x\left(\frac{3}{4}-\frac{1}{6}Y\right)X^3+\\
&+\left(-2YX+\frac{3}{2}(1+x)Y^2-\frac{1}{2}Y-\frac{2}{3}\pi^2x+\frac{2}{3}\pi^2\right)\mathrm{Li}_2(-x)-\frac{1}{12}xX^4-\frac{5}{24}Y^4+\\
&+\left(-\frac{511}{64}+\frac{7}{40}\pi^4+\frac{15}{4}\zeta_3-\frac{29}{48}\pi^2\right)x+\\
&+i\pi\Bigg\{3+\left[(3+x)X\mathrm{Li}_2(-x)-\left(4x+\frac{11}{4}\right)X\right]\frac{x}{y}+\left[\frac{1}{4}X+3X\mathrm{Li}_2(-x)\right]\frac{1}{y}+\\
&\qquad+(4+x)\mathrm{Li}_3(-x)+(3x+2)\mathrm{Li}_3(-y)-\left(\frac{1}{2}-(3x+1)Y\right)\mathrm{Li}_2(-x)-\\
&\qquad-\frac{1}{3}xX^3+\frac{3}{2}xX^2+\Big((1+2x)Y^2-(3x+2)Y\Big)X-\frac{1}{6}Y^3-\\
&\qquad-\left(\left(4+\frac{\pi^2}{2}\right)x+\frac{7}{4}+\frac{2}{3}\pi^2\right)Y+\frac{\pi^2}{6}-2\zeta_3+\left(\frac{y}{4x}-\frac{3}{2}y\right)Y^2+\\
&\qquad+\Bigg[2\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)-\left(1-\frac{1}{2}Y\right)\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y^2-\frac{3}{4}Y\right)X-\frac{1}{12}Y^3-\\
&\qquad\quad-\left(\frac{\pi^2}{3}+2\right)Y-\zeta_3\Bigg]\frac{1}{x}+\frac{1}{xy}X\mathrm{Li}_2(-x)\Bigg\}, \qquad \text{(A.21)}
\end{aligned}
$$

$$
\begin{aligned}
J_2^{[2]} =&-\frac{1}{3}x(\mathrm{Li}_3(-y)+Y\mathrm{Li}_2(-x))-\frac{1}{6}xY^2X+\frac{1}{9}yY^3+\left(\frac{13}{18}+\frac{29}{36}x\right)Y^2+\\
&+\left(\left(-\frac{31}{27}+\frac{5}{24}\pi^2\right)x+\frac{2}{9}\pi^2-\frac{8}{9}\right)Y+\left(\frac{455}{108}+\frac{49}{36}\zeta_3-\frac{25}{72}\pi^2\right)x-\frac{\pi^2}{9}+\\
&+\left(-\frac{1}{18}Y^3+\frac{4}{9}Y^2+\frac{\pi^2}{9}Y\right)\frac{1}{x}-\\
&-i\frac{\pi}{3}\Bigg\{x\mathrm{Li}_2(-x)-yY^2-\left(\frac{13}{3}+\frac{29}{6}x\right)Y+\left(\frac{31}{9}+\frac{1}{24}\pi^2\right)x+\\
&\qquad+\left(\frac{1}{2}Y^2-\frac{8}{3}Y\right)\frac{1}{x}+\frac{8}{3}\Bigg\}, \qquad \text{(A.22)}
\end{aligned}
$$

$$
\begin{aligned}
K_2^{[2]} =&\ \frac{1}{3}x(\mathrm{Li}_3(-x)-\mathrm{Li}_3(-y)-(Y+X)\mathrm{Li}_2(-x))+\frac{1}{9}xX^3-\left(\frac{29}{36}+\frac{1}{6}Y\right)xX^2+\\
&+\left(-\frac{1}{6}Y^2-\frac{11}{72}\pi^2+\frac{31}{27}\right)xX+\frac{1}{9}yY^3+\left(\frac{13}{18}+\frac{29}{36}x\right)Y^2+\\
&+\left(\left(-\frac{31}{27}+\frac{5}{24}\pi^2\right)x+\frac{2}{9}\pi^2-\frac{8}{9}\right)Y+\left(\frac{685}{162}+\frac{35}{36}\zeta_3+\frac{7}{72}\pi^2\right)x+\\
&+\left(-\frac{1}{18}Y^3+\frac{4}{9}Y^2+\frac{\pi^2}{9}Y\right)\frac{1}{x}-\frac{\pi^2}{9}+\\
&+i\frac{\pi}{3}\Bigg\{-2x\mathrm{Li}_2(-x)+xX^2-\left(\frac{29}{6}+Y\right)xX+yY^2+
\end{aligned}
$$

$$+\left(\frac{13}{3}+\frac{29}{6}x\right)Y+\frac{\pi^2}{6}x-\left(\frac{1}{2}Y^2-\frac{8}{3}Y\right)\frac{1}{x}-\frac{8}{3}\Bigg\}\,, \tag{A.23}$$

$$L_2^{[2]} = -\frac{25}{81}x\,, \tag{A.24}$$

For $h=3$ in eq. (2.13) and color factor $\mathrm{Tr}^{[1]}$ in eq. (2.20):

$$\begin{aligned}
A_3^{[1]} = {} & \Bigg[-3\mathrm{Li}_3(-y)-\left(\frac{\pi^2}{2}x+3Y+\frac{\pi^2}{2}\right)\mathrm{Li}_2(-x)-\frac{3}{2}Y^2X-\\
& -\frac{1}{2}Y^3+\pi^2Y+\frac{\pi^2}{2}+3\zeta_3\Bigg]\frac{x}{y}+xY^3X-\frac{1}{4}xY^4+\frac{\pi^4}{8}x-\\
& -3x\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)-Y(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))\right)+\\
& +\Bigg[-\mathrm{Li}_4(-y)-\left(\frac{29}{6}-Y\right)\mathrm{Li}_3(-y)+\left(\frac{1}{2}Y^2-\frac{\pi^2}{3}-\frac{29}{6}Y\right)\mathrm{Li}_2(-x)+\frac{1}{8}Y^4+\\
& \quad+\left(\frac{1}{3}Y^3-\frac{29}{12}Y^2\right)X-\frac{23}{18}Y^3+\left(-\frac{2}{3}\pi^2+\frac{49}{12}\right)Y^2+\frac{413}{72}\zeta_3+\\
& \quad+\left(-\frac{1513}{432}-3\zeta_3+\frac{47}{18}\pi^2\right)Y+\frac{13}{288}\pi^4-\frac{\pi^2}{12}+\frac{23213}{5184}\Bigg]\frac{1}{y}+\frac{3}{2xy}Y^2+\\
& +i\pi\Bigg\{-\left[3\mathrm{Li}_2(-x)+\frac{3}{2}Y^2\right]\frac{x}{y}-3x\left(\frac{\pi^2}{6}Y-\mathrm{Li}_3(-x)-\mathrm{Li}_3(-y)-\frac{1}{2}Y^2X\right)+\\
& \quad+\Bigg[\mathrm{Li}_3(-y)-\left(\frac{29}{6}-Y\right)\mathrm{Li}_2(-x)+\frac{1}{2}Y^2X+\frac{1}{2}Y^3-\frac{23}{6}Y^2-\\
& \quad-\left(\frac{\pi^2}{3}-\frac{49}{6}\right)Y-\frac{1513}{432}-3\zeta_3+\frac{\pi^2}{18}\Bigg]\frac{1}{y}-\frac{1}{2}xY^3+\frac{3}{xy}Y\Bigg\}\,,
\end{aligned} \tag{A.25}$$

$$\begin{aligned}
B_3^{[1]} = {} & \Bigg[-(6+2x)\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)-Y\mathrm{Li}_3(-y)\right)+(3xX+x+6)\mathrm{Li}_3(-y)-\\
& -(4x+6)(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)-Y\mathrm{Li}_3(-x))+(X+1)x\mathrm{Li}_3(-x)+\\
& +\left(-x(1-Y)X-xY^2+(x+6)Y+\frac{\pi^2}{3}x+\pi^2\right)\mathrm{Li}_2(-x)+\frac{1}{12}xX^4-\\
& -\left(\frac{1}{6}xY-\frac{1}{4}+\frac{59}{72}x\right)X^3+\left(2+\frac{1}{6}x\right)Y^3X+\left(\frac{17}{4}+\frac{35}{24}x\right)Y^2X+\\
& +\left(\frac{3}{2}xY^2+\left(\frac{1}{2}x-1\right)Y+\frac{7}{12}+\left(\frac{5}{24}\pi^2+\frac{73}{18}\right)x\right)X^2-\\
& -\left(\frac{61}{12}+\left(\frac{73}{9}+\frac{11}{12}\pi^2\right)x\right)YX+\left(\frac{64}{9}-\frac{71}{72}\pi^2x+\frac{\pi^2}{6}\right)X+\\
& +\left(\frac{9}{2}+\left(\frac{73}{18}+\frac{3}{8}\pi^2\right)x\right)Y^2-\left(\frac{64}{9}+\frac{9}{4}\pi^2+\frac{41}{36}\pi^2x\right)Y+\\
& +\left(\frac{77}{360}\pi^4+\frac{73}{18}\pi^2\right)x+\frac{31}{18}\pi^2+\frac{\pi^4}{4}-2\zeta_3\Bigg]\frac{x}{y}+\\
& +\Bigg[-4\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)+\left(2Y+\frac{8}{3}-X\right)\mathrm{Li}_3(-x)+
\end{aligned}$$

$$
\begin{aligned}
&+\left(Y+\frac{37}{3}-3X\right)\mathrm{Li}_3(-y)+\frac{1}{24}X^4-\left(\frac{1}{3}Y+\frac{65}{72}\right)X^3+\frac{179}{24}Y^2X-\\
&-\left(\left(Y+\frac{8}{3}\right)X-\frac{37}{3}Y+\frac{1}{2}Y^2-\frac{5}{3}\pi^2\right)\mathrm{Li}_2(-x)-\left(\frac{175}{36}-\frac{3}{4}\pi^2\right)YX+\\
&+\left(\frac{5}{24}\pi^2-\frac{1}{2}Y^2-\frac{1}{3}Y+\frac{349}{72}\right)X^2-\left(\zeta_3-\frac{34}{27}+\frac{13}{12}\pi^2\right)X-\frac{1}{6}Y^3X+\\
&+\left(\frac{15}{8}\pi^2-\frac{19}{18}\right)Y^2-\left(\frac{811}{54}-\frac{1}{2}\zeta_3+\frac{263}{48}\pi^2\right)Y-\frac{851}{72}\zeta_3-\frac{53}{288}\pi^4-\\
&-\frac{151}{48}\pi^2+\frac{30659}{1296}\Bigg]\frac{1}{y}-\frac{1}{36}(47-29x)Y^3-\frac{1}{4}yY^4-\frac{7}{2xy}Y^2+\\
&+i\pi\Bigg\{\Bigg[(5x+6)(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))+(xX+6-xY)\mathrm{Li}_2(-x)-\\
&\quad-\left(\frac{1}{4}+\frac{35}{24}x-\frac{1}{2}xY\right)X^2+(2x+3)Y^2X+\left(\frac{35}{12}x+\frac{1}{2}\right)YX+\\
&\quad+\left(\frac{\pi^2}{6}x-\frac{47}{12}\right)X-\left(1+\frac{5}{6}x\right)Y^3+\left(\frac{11}{4}-\frac{35}{24}x\right)Y^2-\\
&\quad-\left(\pi^2-\frac{47}{12}+\frac{\pi^2}{2}x\right)Y-\frac{35}{24}\pi^2x-\frac{\pi^2}{12}\Bigg]\frac{x}{y}+\\
&\quad+\Bigg[\mathrm{Li}_3(-x)-2\mathrm{Li}_3(-y)-\left(X+2Y-\frac{29}{3}\right)\mathrm{Li}_2(-x)-\frac{41}{24}X^2-\frac{7}{6}Y^3+\\
&\qquad+\left(-\frac{3}{2}Y^2+\frac{29}{6}+\frac{23}{12}Y+\frac{\pi^2}{2}\right)X+\frac{125}{24}Y^2+\left(\frac{5}{6}\pi^2-\frac{251}{36}\right)Y-\\
&\qquad-\frac{169}{144}\pi^2-\frac{1}{2}\zeta_3-\frac{743}{54}\Bigg]\frac{1}{y}-\frac{7}{xy}Y\Bigg\},
\end{aligned}\tag{A.26}
$$

$$
\begin{aligned}
C_3^{[1]}=&\Bigg[-6x(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y))+(5xX-4-2xY-3x)\mathrm{Li}_3(-x)-\\
&-(3x+4-4xY+xX)\mathrm{Li}_3(-y)+\frac{1}{12}xX^4+\left(\frac{7}{4}-\frac{7}{24}x-\frac{5}{6}xY\right)X^3-\\
&-(2xX^2-(4+xY+3x)X-Y^2x+(4+3x)Y)\mathrm{Li}_2(-x)+\\
&+\left(\frac{1}{2}xY^2+\left(\frac{5}{4}x-\frac{5}{2}\right)Y+\frac{9}{4}+\left(\frac{5}{24}\pi^2+\frac{3}{2}\right)x\right)X^2-\frac{1}{12}xY^4+\\
&+\left(\frac{5}{6}xY^3-\left(\frac{15}{4}+\left(3+\frac{\pi^2}{4}\right)x\right)Y+\frac{7}{3}\pi^2+\frac{5}{24}\pi^2x+6\right)X+\frac{4}{3}\pi^2+\\
&+\frac{\pi^2}{24}xY^2+\left(\frac{7}{12}\pi^2-6+\frac{2}{3}\pi^2x\right)Y+\left(\frac{3}{2}\pi^2+\frac{19}{40}\pi^4\right)x\Bigg]\frac{x}{y}+\\
&+\Bigg[8\mathrm{Li}_4\left(-\frac{x}{y}\right)+6\mathrm{Li}_4(-x)+2\mathrm{Li}_4(-y)-(1+6Y-3X)\mathrm{Li}_3(-x)-\frac{1}{24}X^4+\\
&\quad+\frac{5}{12}Y^4-(8Y+1-9X)\mathrm{Li}_3(-y)+\left(\frac{3}{4}Y+5Y^2-\frac{9}{8}\pi^2+\frac{65}{8}\right)X^2-\\
&\quad-\left(2X^2-(7Y+1)X+Y+\frac{8}{3}\pi^2+3Y^2\right)\mathrm{Li}_2(-x)-\left(\frac{5}{24}+\frac{2}{3}Y\right)X^3-
\end{aligned}
$$

$$
\begin{aligned}
& \quad -\left(\frac{7}{2}Y^3+\left(\frac{31}{12}\pi^2+\frac{35}{4}\right)Y+6+\frac{\pi^2}{8}+8\zeta_3\right)X-\left(\frac{13}{24}\pi^2-\frac{3}{2}\right)Y^2- \\
& \quad -\left(\frac{93}{16}+\frac{29}{6}\pi^2-11\zeta_3\right)Y+\frac{511}{64}-\frac{15}{4}\zeta_3+\frac{169}{360}\pi^4+\frac{17}{16}\pi^2\bigg]\frac{1}{y}- \\
& -\frac{1}{8}(11-13x)Y^2X+\frac{1}{3}xY^3-\frac{3}{2}xY^2+\frac{3}{2xy}Y^2+ \\
& +i\pi\Bigg\{\Bigg[3x(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+(Y-X)\mathrm{Li}_2(-x))-\left(\frac{9}{8}x+\frac{3}{2}xY-\frac{3}{4}\right)X^2+ \\
& \qquad +\left(3xY^2-\left(\frac{3}{2}-\frac{9}{4}x\right)Y+\frac{3}{4}-\frac{\pi^2}{2}x\right)X-\left(\frac{3}{4}+\frac{3}{2}\pi^2x\right)Y- \\
& \qquad -\frac{9}{8}\pi^2x+\frac{\pi^2}{4}\Bigg]\frac{x}{y}+ \\
& \qquad +\Bigg[-3\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+(Y+3X)\mathrm{Li}_2(-x)-\frac{3}{8}X^2+ \\
& \qquad\quad +\left(2Y^2+\frac{15}{2}+\frac{21}{4}Y-\frac{3}{2}\pi^2\right)X-\left(\frac{23}{4}+\frac{2}{3}\pi^2\right)Y-\frac{9}{8}\pi^2- \\
& \qquad\quad -\frac{189}{16}+3\zeta_3\Bigg]\frac{1}{y}+\frac{1}{2}(x-1)Y^3-\frac{1}{8}(15-9x)Y^2+\frac{3}{xy}Y\Bigg\}\,,
\end{aligned}
\tag{A.27}
$$

$$
\begin{aligned}
D_3^{[1]} &= \left[\frac{1}{3}\mathrm{Li}_3(-y)+\frac{1}{3}Y\mathrm{Li}_2(-x)+\frac{1}{6}Y^2X+\frac{5}{18}Y^3-\frac{37}{36}Y^2-\left(\frac{11}{18}\pi^2-\frac{145}{54}\right)Y+\right. \\
& \quad \left.+\frac{41}{72}\pi^2-\frac{49}{36}\zeta_3-\frac{455}{108}\right]\frac{1}{y}-i\frac{\pi}{3y}\left\{\frac{\pi^2}{6}-\frac{5}{2}Y^2+\frac{37}{6}Y-\mathrm{Li}_2(-x)-\frac{145}{18}\right\},
\end{aligned}
\tag{A.28}
$$

$$
\begin{aligned}
E_3^{[1]} &= \Bigg[\frac{1}{9}xX^3-\left(\frac{8}{9}x+\frac{1}{3}\right)X^2+\left(-\frac{1}{3}xY^2+\left(\frac{4}{3}+\frac{16}{9}x\right)Y+\frac{\pi^2}{9}x-\frac{16}{9}\right)X+ \\
& \quad +\frac{1}{9}(2\pi^2x+16)Y-\frac{8}{9}\pi^2x-\frac{5}{9}\pi^2\Bigg]\frac{x}{y}+ \\
& +\Bigg[-\frac{2}{3}\mathrm{Li}_3(-x)-\frac{4}{3}\mathrm{Li}_3(-y)-\left(\frac{4}{3}Y-\frac{2}{3}X\right)\mathrm{Li}_2(-x)+\frac{1}{9}X^3+\left(\frac{1}{3}Y-\frac{19}{18}\right)X^2+ \\
& \quad +\left(\frac{14}{27}-\frac{4}{3}Y^2+\frac{7}{9}Y+\frac{\pi^2}{4}\right)X+\left(\frac{41}{24}\pi^2+\frac{107}{27}\right)Y-\frac{685}{162}- \\
& \quad -\frac{11}{36}\zeta_3+\frac{5}{8}\pi^2\Bigg]\frac{1}{y}+\frac{2}{9}(1-x)Y^3+\frac{1}{9}(1+8x)Y^2+ \\
& +i\frac{\pi}{3}\Bigg\{\left[xX^2+2(1-xY)X-2Y+\pi^2x+xY^2\right]\frac{x}{y}+ \\
& \qquad +\left[-2\mathrm{Li}_2(-x)+X^2-(2Y+4)X+\frac{121}{9}+\frac{29}{24}\pi^2+\frac{5}{3}Y-4Y^2\right]\frac{1}{y}\Bigg\}\,,
\end{aligned}
\tag{A.29}
$$

$$
F_3^{[1]} = \left[\frac{1}{9}Y^2-\frac{10}{27}Y-\frac{\pi^2}{9}+\frac{25}{81}\right]\frac{1}{y}+i\pi\left[\frac{2}{9}Y-\frac{10}{27}\right]\frac{1}{y}\,,
\tag{A.30}
$$

For $h=3$ in eq. (2.13) and color factor $\mathrm{Tr}^{[2]}$ in eq. (2.20):

$$
\begin{aligned}
G_3^{[2]} = & \Bigg[\left(3-\frac{1}{2}x\right)\mathrm{Li}_4\left(-\frac{x}{y}\right)-\Big(2xX+(3+x)Y\Big)\mathrm{Li}_3(-y)-\frac{1}{48}xX^4- \\
& -\left(2xX+(3+x)Y-\frac{3}{2}(1-x)\right)\mathrm{Li}_3(-x)+\left(\frac{1}{12}(1+xY)+\frac{11}{36}x\right)X^3+ \\
& +\left(\frac{1}{4}xX^2-\left(\frac{1}{2}xY+\frac{3}{2}(1-x)\right)X+\frac{1}{4}xY^2-\frac{\pi^2}{2}+\frac{\pi^2}{12}x\right)\mathrm{Li}_2(-x)- \\
& -\left(xY^2+\frac{1}{2}Y-\frac{7}{12}+\left(\frac{37}{36}+\frac{5}{24}\pi^2\right)x\right)X^2+\left(\frac{1}{4}+\frac{1}{48}x\right)Y^4-\frac{1}{2}\zeta_3- \\
& -\left(1-\frac{1}{6}x\right)Y^3X-\left(\frac{11}{12}x+\frac{3}{4}\right)Y^2X+\left(-\frac{5}{6}+\left(\frac{3}{4}\pi^2+\frac{37}{18}\right)x\right)YX+ \\
& +\left(-\frac{32}{9}-\frac{\pi^2}{4}+\frac{5}{9}\pi^2x\right)X+\left(\frac{1}{9}x-\frac{1}{3}\right)Y^3+\left(\frac{1}{4}-\left(\frac{37}{36}+\frac{\pi^2}{4}\right)x\right)Y^2+ \\
& +\left(\frac{11}{18}\pi^2x+\frac{5}{12}\pi^2+\frac{32}{9}\right)Y-\left(\frac{157}{720}\pi^4+\frac{37}{36}\pi^2\right)x+\frac{7}{18}\pi^2-\frac{\pi^4}{8}\Bigg]\frac{x}{y}+ \\
& +\Bigg[\frac{5}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)+4\mathrm{Li}_4(-x)-2\mathrm{Li}_4(-y)-\left(\frac{1}{3}+2Y+X\right)\mathrm{Li}_3(-x)-\frac{1}{48}X^4- \\
& \quad -\left(\frac{29}{6}-2X+Y\right)\mathrm{Li}_3(-y)+\left(\frac{5}{12}Y+\frac{5}{8}Y^2+\frac{\pi^2}{4}-\frac{125}{72}\right)X^2+ \\
& \quad +\left(\frac{1}{4}X^2+\left(\frac{1}{3}+\frac{3}{2}Y\right)X-\frac{29}{6}Y-\frac{3}{4}Y^2-\frac{5}{12}\pi^2\right)\mathrm{Li}_2(-x)+\frac{7}{18}X^3- \\
& \quad -\left(\frac{5}{12}Y^3+\frac{17}{4}Y^2-\left(\frac{\pi^2}{12}-\frac{5}{72}\right)Y-\frac{3}{16}\pi^2+\frac{17}{27}-\frac{1}{2}\zeta_3\right)X+\frac{1}{16}Y^4+ \\
& \quad +\frac{1}{3}Y^3-\left(\frac{29}{36}+\frac{13}{24}\pi^2\right)Y^2+\left(\frac{43}{18}\pi^2+3\zeta_3+\frac{3049}{432}\right)Y- \\
& \quad -\frac{43}{1440}\pi^4+\frac{7}{72}\zeta_3+\frac{25}{9}\pi^2-\frac{23213}{5184}\Bigg]\frac{1}{y}+ \\
& +\frac{1}{2xy}Y^2-3x(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y))+\frac{3}{2}x(\mathrm{Li}_3(-y)+Y\mathrm{Li}_2(-x))+ \\
& +i\pi\Bigg\{\Bigg[-3\mathrm{Li}_2(-x)+\frac{1}{6}xX^3+\left(\frac{1}{2}+\frac{1}{3}x\right)Y^3-\left(\frac{1}{2}xY-\frac{1}{2}-\frac{1}{6}x\right)X^2- \\
& \qquad -\left(Y^2+\left(\frac{1}{3}x+1\right)Y-\frac{\pi^2}{6}x-\frac{1}{3}\right)X+\left(\frac{\pi^2}{2}+\frac{\pi^2}{3}x-\frac{1}{3}\right)Y- \\
& \qquad -\left(1-\frac{1}{6}x\right)Y^2\Bigg]\frac{x}{y}+3x(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))-\frac{\pi^2}{6}x+ \\
& \qquad +\Bigg[-3\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+\left(2X-\frac{9}{2}\right)\mathrm{Li}_2(-x)-\frac{1}{12}X^3+\frac{1}{4}Y^3+ \\
& \qquad\quad +\left(\frac{17}{12}-\frac{1}{2}Y\right)X^2+\left(\frac{1}{2}Y^2-\frac{85}{24}-\frac{17}{6}Y+\frac{\pi^2}{4}\right)X-\frac{5}{6}Y^2+\frac{41}{48}\pi^2- \\
& \qquad\quad -\left(\frac{\pi^2}{4}+\frac{121}{72}\right)Y+\frac{7}{2}\zeta_3+\frac{2777}{432}\Bigg]\frac{1}{y}+\left(x-\frac{1}{2}\right)Y^2X+\frac{1}{xy}Y\Bigg\}, \qquad \text{(A.31)}
\end{aligned}
$$

$$
\begin{aligned}
H_3^{[2]} = {} & \Bigg[2x(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)) + \left(1 + xY - 2xX + \frac{5}{2}x\right)\mathrm{Li}_3(-x) - \frac{1}{8}xX^4 + \\
& + \left(\frac{5}{2}x - xY + 1\right)\mathrm{Li}_3(-y) - \left(\frac{17}{12} - \frac{55}{72}x - \frac{2}{3}xY\right)X^3 - \left(\frac{7}{4} - \frac{17}{24}x\right)Y^2X + \\
& + \frac{1}{24}xY^4 + \left(xX^2 - \left(\frac{5}{2}x + xY + 1\right)X + \left(1 + \frac{5}{2}x\right)Y\right)\mathrm{Li}_2(-x) + \\
& + \left(\left(3 - \frac{3}{2}x\right)Y - \frac{13}{6} - \left(\frac{\pi^2}{8} + \frac{127}{36} + \frac{3}{4}Y^2\right)x\right)X^2 - \left(\frac{31}{180}\pi^4 + \frac{127}{36}\pi^2\right)x + \\
& + \left(\left(\frac{89}{12} + \left(\frac{\pi^2}{4} + \frac{127}{18}\right)x\right)Y - \frac{59}{9} + \frac{25}{72}\pi^2 x - \frac{4}{3}\pi^2\right)X - \\
& - \left(\frac{21}{4} + \left(\frac{\pi^2}{8} + \frac{127}{36}\right)x\right)Y^2 + \left(\frac{5}{12}\pi^2 + \frac{59}{9} + \frac{\pi^2}{36}x\right)Y - \frac{28}{9}\pi^2\Bigg]\frac{x}{y} + \\
& + \Bigg[-4\mathrm{Li}_4\left(-\frac{x}{y}\right) - 6\mathrm{Li}_4(-x) + \mathrm{Li}_4(-y) + \left(4Y - \frac{11}{6}\right)\mathrm{Li}_3(-x) - \frac{5}{24}Y^4 - \\
& \quad - \left(5X - 4Y + \frac{13}{6}\right)\mathrm{Li}_3(-y) - \left(\frac{7}{12}Y + \frac{283}{36} + \frac{27}{8}Y^2 - \frac{\pi^2}{4}\right)X^2 + \\
& \quad + \left(X^2 - \left(5Y - \frac{11}{6}\right)X + \pi^2 + \frac{5}{2}Y^2 - \frac{13}{6}Y\right)\mathrm{Li}_2(-x) + \left(\frac{3}{4}Y + \frac{49}{72}\right)X^3 + \\
& \quad + \left(\frac{29}{12}Y^3 - \frac{43}{24}Y^2 + \left(\frac{11}{12}\pi^2 + \frac{679}{72}\right)Y + \frac{11}{2}\zeta_3 + \frac{17}{16}\pi^2 + \frac{64}{27}\right)X - \\
& \quad - \left(\frac{\pi^2}{4} + \frac{29}{9}\right)Y^2 + \left(\frac{57}{16}\pi^2 + \frac{781}{54} - \frac{11}{2}\zeta_3\right)Y - \frac{30659}{1296} + \frac{467}{72}\zeta_3 + \\
& \quad + \frac{101}{144}\pi^2 - \frac{263}{1440}\pi^4\Bigg]\frac{1}{y} + \frac{1}{36}(13 - 31x)Y^3 + \frac{1}{2xy}Y^2 + \\
& + i\pi\Bigg\{\Bigg[-x(\mathrm{Li}_3(-x) + \mathrm{Li}_3(-y) - (X - Y)\mathrm{Li}_2(-x)) - \frac{1}{6}xX^3 + \frac{1}{3}xY^3 + \\
& \qquad + \left(\frac{49}{24}x + xY - \frac{3}{4}\right)X^2 + \left(\frac{37}{12} - \frac{3}{2}xY^2 + \left(\frac{3}{2} - \frac{49}{12}x\right)Y\right)X + \\
& \qquad + \left(\frac{49}{24}x - \frac{3}{4}\right)Y^2 + \left(\frac{2}{3}\pi^2 x - \frac{37}{12}\right)Y + \frac{49}{24}\pi^2 x - \frac{\pi^2}{4}\Bigg]\frac{x}{y} + \\
& \qquad + \Bigg[4\mathrm{Li}_3(-x) - \mathrm{Li}_3(-y) - \left(3X + \frac{1}{3}\right)\mathrm{Li}_2(-x) + \frac{1}{12}X^3 + \frac{1}{12}Y^3 + \\
& \qquad\quad + \left(\frac{13}{24} + \frac{1}{2}Y\right)X^2 - \left(\frac{3}{2}Y^2 + \frac{151}{24} + \frac{31}{12}Y - \frac{\pi^2}{4}\right)X - \frac{43}{24}Y^2 + \\
& \qquad\quad + \left(\frac{\pi^2}{4} + \frac{215}{72}\right)Y + \frac{101}{6} + \frac{61}{72}\pi^2\Bigg]\frac{1}{y} + \frac{1}{xy}Y\Bigg\}, \qquad \text{(A.32)}
\end{aligned}
$$

$$
\begin{aligned}
I_3^{[2]} = {} & \Bigg[-\frac{1}{2}x\mathrm{Li}_4\left(-\frac{x}{y}\right) + 2x\mathrm{Li}_4(-x) + 2x\mathrm{Li}_4(-y) + \left(\frac{3}{2} + xY + x - 2xX\right) \times \\
& \times \mathrm{Li}_3(-x) + \left(x - xY + \frac{3}{2}\right)\mathrm{Li}_3(-y) - \left(\frac{2}{3} - \frac{1}{4}xY - \frac{1}{24}x\right)X^3 +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(1-\frac{1}{4}x(1+Y)\right)YX^2+\\
&+\left(\frac{3}{4}xX^2-\left(\frac{3}{2}+\frac{1}{2}xY+x\right)X-\frac{1}{4}xY^2+\left(\frac{3}{2}+x\right)Y+\frac{\pi^2}{12}x\right)\mathrm{Li}_2(-x)-\\
&-\left(\frac{5}{4}+\left(\frac{\pi^2}{12}+1\right)x\right)X^2+\frac{1}{48}xY^4-\frac{1}{6}xY^3X+\left(\frac{9}{4}+\left(\frac{\pi^2}{6}+2\right)x\right)YX-\\
&-\left(\frac{\pi^2}{8}x+\frac{11}{12}\pi^2+3\right)X-\left(1+\left(\frac{\pi^2}{24}+1\right)x\right)Y^2-\left(\frac{\pi^2}{6}x-3+\frac{\pi^2}{6}\right)Y-\\
&-\left(\pi^2+\frac{25}{144}\pi^4\right)x-\frac{5}{6}\pi^2-\frac{1}{2}\zeta_3\Big]\frac{x}{y}+\\
&+\Big[-\frac{5}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)-\mathrm{Li}_4(-y)-\frac{7}{48}Y^4+\left(2Y+\frac{1}{2}-X\right)\mathrm{Li}_3(-x)+\\
&\quad+\left(\frac{1}{2}-3X+3Y\right)\mathrm{Li}_3(-y)-\left(\frac{1}{4}Y+\frac{7}{4}Y^2+\frac{27}{8}-\frac{5}{12}\pi^2\right)X^2+\frac{4}{3}Y^3X+\\
&\quad+\left(\frac{3}{4}X^2-\frac{1}{2}(1+5Y)X+\frac{5}{4}Y^2+\frac{3}{4}\pi^2+\frac{1}{2}Y\right)\mathrm{Li}_2(-x)+\left(\frac{1}{24}+\frac{1}{4}Y\right)X^3+\\
&\quad+\left(\left(\frac{17}{4}+\frac{5}{6}\pi^2\right)Y+3\zeta_3+3-\frac{\pi^2}{24}\right)X+\left(\frac{\pi^2}{8}-\frac{1}{2}\right)Y^2+\\
&\quad+\left(-6\zeta_3+\frac{7}{4}\pi^2+\frac{45}{16}\right)Y-\frac{17}{144}\pi^4-\frac{511}{64}-\frac{21}{16}\pi^2+\frac{13}{4}\zeta_3\Big]\frac{1}{y}-\\
&-\frac{1}{48}(1-x)X^4+\frac{3}{8}(1-x)Y^2X-\frac{1}{6}xY^3-\frac{1}{2xy}Y^2+\\
&+i\pi\Big\{\Big[-x\mathrm{Li}_3(-x)-x\mathrm{Li}_3(-y)+x(X-Y)\mathrm{Li}_2(-x)+\left(\frac{3}{8}x+\frac{1}{2}xY-\frac{1}{4}\right)X^2-\\
&\qquad\quad-\Big(xY^2-\left(\frac{1}{2}-\frac{3}{4}x\right)Y+\\
&\qquad+\frac{1}{4}-\frac{\pi^2}{6}x\Big)X+\left(\frac{1}{4}+\frac{\pi^2}{2}x\right)Y-\frac{\pi^2}{12}+\frac{3}{8}\pi^2x\Big]\frac{x}{y}+\\
&\qquad+\Big[\mathrm{Li}_3(-x)-X\mathrm{Li}_2(-x)+\frac{1}{8}X^2-\left(\frac{5}{2}+\frac{7}{4}Y-\frac{\pi^2}{2}+\frac{1}{2}Y^2\right)X+\\
&\qquad\quad+\left(\frac{13}{4}+\frac{\pi^2}{6}\right)Y+\frac{93}{16}+\frac{13}{24}\pi^2-3\zeta_3\Big]\frac{1}{y}+\\
&\qquad+\frac{1}{6}(1-x)Y^3+\frac{1}{8}(5-3x)Y^2-\frac{1}{yx}Y\Big\}, \qquad\qquad (A.33)
\end{aligned}
$$

$$
\begin{aligned}
J_3^{[2]}=&\Big[-\frac{1}{18}xX^3+\left(\frac{4}{9}x+\frac{1}{6}\right)X^2+\left(\frac{1}{6}xY^2-\left(\frac{8}{9}x+\frac{2}{3}\right)Y+\frac{8}{9}-\frac{\pi^2}{18}x\right)X-\\
&-\frac{1}{9}xY^3+\left(\frac{1}{2}+\frac{4}{9}x\right)Y^2-\frac{1}{9}(\pi^2x+8)Y+\frac{4}{9}\pi^2x+\frac{5}{18}\pi^2\Big]\frac{x}{y}+\\
&+\Big[\frac{1}{3}(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+(Y-X)\mathrm{Li}_2(-x))-\frac{1}{18}X^3-\left(\frac{1}{6}Y-\frac{19}{36}\right)X^2+
\end{aligned}
$$

$$+\left(\frac{1}{2}Y^2-\frac{\pi^2}{8}-\frac{7}{27}-\frac{7}{18}Y\right)X-\frac{1}{6}Y^3+\frac{13}{12}Y^2-\left(\frac{2}{9}\pi^2+\frac{193}{54}\right)Y-$$
$$-\frac{67}{72}\pi^2+\frac{37}{36}\zeta_3+\frac{455}{108}\bigg]\frac{1}{y}+$$
$$+i\pi\bigg\{\left[-\frac{1}{6}xX^2-\frac{1}{3}(1-xY)X-\frac{\pi^2}{6}x-\frac{1}{6}xY^2+\frac{1}{3}Y\right]\frac{x}{y}+$$
$$+\left[-\frac{1}{6}X^2+\frac{1}{3}(2+Y)X-\frac{\pi^2}{8}-\frac{23}{6}-\frac{1}{6}Y^2+\frac{16}{9}Y\right]\frac{1}{y}\bigg\}, \tag{A.34}$$

$$K_3^{[2]}=\bigg[-\frac{1}{18}xX^3+\left(\frac{4}{9}x+\frac{1}{6}\right)X^2+\left(\frac{1}{6}xY^2-\left(\frac{8}{9}x+\frac{2}{3}\right)Y+\frac{8}{9}-\frac{\pi^2}{18}x\right)X-$$
$$-\frac{1}{9}(\pi^2x+8)Y+\frac{4}{9}\pi^2x+\frac{5}{18}\pi^2\bigg]\frac{x}{y}+$$
$$+\bigg[\frac{1}{3}(\mathrm{Li}_3(-x)+2\mathrm{Li}_3(-y)+(2Y-X)\mathrm{Li}_2(-x))-\frac{1}{18}X^3-\left(\frac{1}{6}Y-\frac{19}{36}\right)X^2-$$
$$-\left(\frac{7}{27}-\frac{2}{3}Y^2+\frac{7}{18}Y+\frac{\pi^2}{8}\right)X-\left(\frac{7}{8}\pi^2+\frac{83}{27}\right)Y+\frac{685}{162}+$$
$$+\frac{23}{36}\zeta_3-\frac{19}{72}\pi^2\bigg]\frac{1}{y}+\frac{1}{9}(x-1)Y^3-\left(\frac{1}{18}+\frac{4}{9}x\right)Y^2+$$
$$+i\pi\bigg\{\left[-\frac{1}{6}xX^2-\frac{1}{3}(1-xY)X-\frac{\pi^2}{6}x-\frac{1}{6}xY^2+\frac{1}{3}Y\right]\frac{x}{y}+$$
$$+\left[\mathrm{Li}_2(-x)-\frac{1}{2}X^2+(2+Y)X-\frac{5}{6}Y-\frac{2}{3}\pi^2-10+2Y^2\right]\frac{1}{3y}\bigg\}, \tag{A.35}$$

$$L_3^{[2]}=\left[\frac{\pi^2}{9}-\frac{1}{9}Y^2+\frac{10}{27}Y-\frac{25}{81}\right]\frac{1}{y}-i\pi\left\{\frac{2}{9}Y-\frac{10}{27}\right\}\frac{1}{y}, \tag{A.36}$$

For $h=4$ in eq. (2.14) and color factor $\mathrm{Tr}^{[1]}$ in eq. (2.20):

$$A_4^{[1]}=\bigg[\frac{1}{2}\mathrm{Li}_4(-y)-\frac{4}{3}\mathrm{Li}_3(-y)-\left(\frac{\pi^2}{3}+\frac{4}{3}Y-\frac{1}{4}Y^2\right)\mathrm{Li}_2(-x)-\left(\frac{2}{3}Y^2-\frac{1}{12}Y^3\right)X+$$
$$+\frac{1}{12}Y^4-\frac{5}{8}Y^3+\left(\frac{101}{36}-\frac{3}{8}\pi^2\right)Y^2+\left(-2\zeta_3+\frac{25}{18}\pi^2-\frac{3049}{432}\right)Y+$$
$$+\frac{23213}{5184}-\frac{25}{18}\pi^2+\frac{269}{72}\zeta_3+\frac{41}{1440}\pi^4\bigg]\frac{x}{y}+$$
$$+\bigg[\frac{1}{2}\mathrm{Li}_3(-y)+\frac{1}{2}Y\mathrm{Li}_2(-x)+\frac{1}{4}Y^2X-\frac{1}{4}Y^3+2Y^2+\left(\frac{7}{12}\pi^2-\frac{32}{9}\right)Y-$$
$$-\frac{29}{36}\pi^2+\zeta_3\bigg]\frac{1}{y}+$$
$$+\bigg[-\frac{3}{2}\mathrm{Li}_4(-y)+Y\mathrm{Li}_3(-y)+\frac{1}{4}Y^2\mathrm{Li}_2(-x)+\frac{1}{4}Y^3X+\frac{1}{24}Y^4-\frac{29}{72}Y^3+$$
$$+\left(\frac{16}{9}-\frac{7}{24}\pi^2\right)Y^2+\left(\frac{29}{36}\pi^2-\zeta_3\right)Y+\frac{\pi^4}{60}\bigg]\frac{1}{xy}+$$

$$
\begin{aligned}
&+ i\pi\Bigg\{ \Bigg[\left(\frac{1}{2}Y - \frac{4}{3}\right)\mathrm{Li}_2(-x) + \frac{1}{3}Y^3 - \frac{15}{8}Y^2 + \left(\frac{101}{18} - \frac{\pi^2}{12}\right)Y + \frac{5}{36}\pi^2 - \\
&\qquad - 2\zeta_3 - \frac{3049}{432}\Bigg]\frac{x}{y} + \left[\frac{\pi^2}{12} - \frac{32}{9} - \frac{3}{4}Y^2 + 4Y + \frac{1}{2}\mathrm{Li}_2(-x)\right]\frac{1}{y} + \\
&\quad + \Bigg[\mathrm{Li}_3(-y) + \frac{1}{2}Y\mathrm{Li}_2(-x) + \frac{1}{2}Y^2X + \frac{1}{6}Y^3 - \frac{29}{24}Y^2 - \\
&\qquad - \left(\frac{\pi^2}{4} - \frac{32}{9}\right)Y - \zeta_3\Bigg]\frac{1}{xy}\Bigg\},
\end{aligned} \tag{A.37}
$$

$$
\begin{aligned}
B_4^{[1]} = &\Bigg[2\mathrm{Li}_4\left(-\frac{x}{y}\right) + 2\mathrm{Li}_4(-x) + \frac{5}{2}\mathrm{Li}_4(-y) - \left(2Y - \frac{29}{3}\right)\mathrm{Li}_3(-x) + \frac{1}{8}X^4 - \\
&- \left(4Y - X - \frac{77}{6}\right)\mathrm{Li}_3(-y) + \left(\left(Y - \frac{29}{3}\right)X - \frac{7}{4}Y^2 + \frac{\pi^2}{3} + \frac{77}{6}Y\right)\mathrm{Li}_2(-x) - \\
&- \left(\frac{71}{36} + \frac{1}{2}Y\right)X^3 + \left(\frac{779}{72} + \frac{3}{2}Y^2 - \frac{11}{6}Y + \frac{5}{2}x + \frac{5}{12}\pi^2\right)X^2 - \frac{29}{12}Y^3X + \\
&+ \frac{265}{24}Y^2X - \left(\frac{473}{36} + 5x + \frac{\pi^2}{2}\right)YX - \left(2\zeta_3 + \frac{158}{27} + \frac{245}{72}\pi^2\right)X + \\
&+ \left(\frac{5}{2}x + \frac{17}{12}\pi^2 + \frac{73}{18}\right)Y^2 + \left(\frac{3}{2}\zeta_3 - \frac{205}{54} - \frac{1177}{144}\pi^2\right)Y - \frac{707}{72}\zeta_3 + \\
&+ \frac{30659}{1296} + \frac{583}{144}\pi^2 + \frac{5}{2}\pi^2x - \frac{103}{480}\pi^4\Bigg]\frac{x}{y} - \frac{5}{24}Y^4 + \frac{47}{36}Y^3 + \\
&+ \Bigg[-6\left(\mathrm{Li}_4\left(-\frac{x}{y}\right) + \mathrm{Li}_4(-x) + \mathrm{Li}_4(-y) - \mathrm{Li}_3(-x) + (X - Y)\mathrm{Li}_3(-y)\right) + \\
&\quad + \frac{9}{2}\mathrm{Li}_3(-y) + \left(\pi^2 - 6X + \frac{9}{2}Y\right)\mathrm{Li}_2(-x) - \left(\frac{3}{2}Y^2 + 3Y\right)X^2 + \\
&\quad + Y^3X + \frac{11}{2}Y^2X + \left(2\pi^2 - \frac{1}{4}\right)YX + \left(6\zeta_3 - \frac{7}{6}\pi^2\right)X - \left(2 - \frac{\pi^2}{2}\right)Y^2 + \\
&\quad + \left(\frac{37}{9} - 6\zeta_3 - \frac{13}{3}\pi^2\right)Y + \frac{1}{15}\pi^4 + \frac{85}{36}\pi^2 + \zeta_3\Bigg]\frac{1}{y} - \left(\frac{23}{36}Y^3 - \frac{5}{24}Y^4\right)\frac{1}{x} + \\
&+ \Bigg[-8\mathrm{Li}_4\left(-\frac{x}{y}\right) - 8\mathrm{Li}_4(-x) - \frac{11}{2}\mathrm{Li}_4(-y) + 2Y\mathrm{Li}_3(-x) - 2X^2Y^2 + \frac{17}{12}Y^3X + \\
&\quad + (7Y - 7X - 1)\mathrm{Li}_3(-y) - \left(Y - \frac{1}{4}Y^2 - \frac{5}{3}\pi^2 + YX\right)\mathrm{Li}_2(-x) + \frac{7}{3}\pi^2YX - \\
&\quad - \frac{7}{8}Y^2X - \left(\frac{14}{9} - \frac{4}{3}\pi^2\right)Y^2 + 7\zeta_3X - \left(\frac{19}{36}\pi^2 + 7\zeta_3\right)Y + \zeta_3 + \frac{11}{180}\pi^4\Bigg]\frac{1}{xy} + \\
&+ i\pi\Bigg\{\Bigg[-2\mathrm{Li}_3(-x) - 3\mathrm{Li}_3(-y) - \left(\frac{5}{2}Y - \frac{19}{6} - X\right)\mathrm{Li}_2(-x) - \left(\frac{35}{12} - \frac{1}{2}Y\right)X^2 + \\
&\qquad + \left(\frac{2}{3}\pi^2 + \frac{17}{2} - 2Y^2 + \frac{67}{12}Y\right)X - \frac{1}{3}Y^3 + \frac{17}{24}Y^2 + \left(\pi^2 - \frac{181}{36}\right)Y - \\
&\qquad - \frac{521}{54} - \frac{415}{144}\pi^2 - \frac{1}{2}\zeta_3\Bigg]\frac{x}{y} +
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{3}{2}\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y-\frac{1}{4}\right)X-\frac{\pi^2}{3}+\frac{5}{4}Y^2-\frac{17}{4}Y+\frac{37}{9}\right]\frac{1}{y}+\\
&+\left[2\mathrm{Li}_3(-x)-\left(X+\frac{1}{2}Y+1\right)\mathrm{Li}_2(-x)-\left(\frac{3}{4}Y+\frac{1}{2}Y^2\right)X-\frac{2}{3}Y^3+\right.\\
&\quad\left.+\frac{37}{24}Y^2-\left(\frac{28}{9}-\frac{\pi^2}{3}\right)Y\right]\frac{1}{xy}\Bigg\},
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
C_4^{[1]} =\ &\left[2\mathrm{Li}_4\left(-\frac{x}{y}\right)-6\mathrm{Li}_4(-x)-10\mathrm{Li}_4(-y)-(2Y-8X)\mathrm{Li}_3(-x)-\left(\frac{3}{2}Y+\frac{9}{4}\right)X^3+\right.\\
&+\frac{1}{24}X^4+(5X+4Y-1)\mathrm{Li}_3(-y)-\left(Y+\frac{2}{3}\pi^2-5YX+4X^2\right)\mathrm{Li}_2(-x)+\\
&+\left(\frac{3}{2}x-\frac{11}{12}\pi^2+4Y^2+\frac{9}{2}Y+\frac{71}{8}\right)X^2-\frac{1}{6}Y^3X-\left(\frac{11}{6}\pi^2+\frac{41}{4}+3x\right)YX-\\
&-\left(\frac{7}{4}\pi^2+12+5\zeta_3\right)X+\left(\frac{3}{2}x+\frac{\pi^2}{8}+\frac{3}{2}\right)Y^2+\left(3\zeta_3+\frac{99}{16}-\frac{13}{12}\pi^2\right)Y+\\
&\left.+\frac{91}{90}\pi^4-\frac{35}{4}\zeta_3+\frac{511}{64}+\frac{151}{48}\pi^2+\frac{3}{2}\pi^2x\right]\frac{x}{y}+\\
&+\left[2\mathrm{Li}_3(-y)+2Y\mathrm{Li}_2(-x)+\left(\frac{\pi^2}{2}+\frac{3}{4}Y\right)X-\frac{9}{2}Y^2+\left(\frac{3}{4}\pi^2+6\right)Y+\right.\\
&\quad\left.+\frac{5}{12}\pi^2-8\zeta_3\right]\frac{1}{y}+\frac{19}{8}Y^2X-\left(\frac{1}{3}Y^3+\frac{21}{8}Y^2X\right)\frac{1}{x}+\\
&+\left[6\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)-Y\mathrm{Li}_3(-x)\right)-(8Y-3X-3)\mathrm{Li}_3(-y)-\right.\\
&\quad-(2Y^2-3Y-3YX+2\pi^2)\mathrm{Li}_2(-x)+\frac{3}{2}X^2Y^2-\left(3\zeta_3+\frac{5}{2}Y^3+\pi^2Y\right)X+\\
&\quad\left.+\frac{1}{3}Y^4-\left(\frac{5}{8}\pi^2+\frac{9}{2}\right)Y^2-\left(\frac{35}{12}\pi^2-8\zeta_3\right)Y-3\zeta_3-\frac{\pi^4}{15}\right]\frac{1}{xy}+\\
&+i\pi\Bigg\{\left[6\mathrm{Li}_3(-x)+9\mathrm{Li}_3(-y)+(5Y-1-3X)\mathrm{Li}_2(-x)-\left(\frac{3}{2}Y+\frac{9}{4}\right)X^2-2\zeta_3+\right.\\
&\qquad\left.+\left(6Y^2+\frac{15}{2}-2\pi^2+\frac{21}{4}Y\right)X-\left(\frac{31}{12}\pi^2+\frac{29}{4}\right)Y-\frac{93}{16}-\frac{25}{12}\pi^2\right]\frac{x}{y}+\\
&\quad+\left[2\mathrm{Li}_2(-x)-\left(\frac{3}{2}Y-\frac{3}{4}\right)X+\frac{5}{12}\pi^2-\frac{33}{4}Y+6\right]\frac{1}{y}+\frac{5}{6}Y^3+\frac{15}{8}Y^2+\\
&\quad+\left[-6\mathrm{Li}_3(-x)-5\mathrm{Li}_3(-y)-(Y-3X-3)\mathrm{Li}_2(-x)+\left(\frac{9}{4}Y-Y^2\right)X-\right.\\
&\qquad\left.-\left(9-\frac{5}{12}\pi^2\right)Y+5\zeta_3\right]\frac{1}{xy}-\left(\frac{17}{8}Y^2+\frac{5}{6}Y^3\right)\frac{1}{x}\Bigg\},
\end{aligned}
\tag{A.39}
$$

$$
\begin{aligned}
D_4^{[1]} =\ &\left[\frac{1}{3}\mathrm{Li}_3(-y)+\frac{1}{3}Y\mathrm{Li}_2(-x)+\frac{1}{6}Y^2X+\frac{1}{6}Y^3-\frac{13}{12}Y^2-\left(\frac{7}{18}\pi^2-\frac{193}{54}\right)Y-\frac{455}{108}+\right.\\
&\left.+\frac{19}{24}\pi^2-\frac{49}{36}\zeta_3\right]\frac{x}{y}+\left[\frac{2}{9}\pi^2-\frac{1}{2}Y^2+\frac{8}{9}Y\right]\frac{1}{y}+\left[\frac{1}{9}Y^3-\frac{4}{9}Y^2-\frac{2}{9}\pi^2Y\right]\frac{1}{xy}+
\end{aligned}
$$

$$
\begin{aligned}
&+ i\pi\Bigg\{\left[\frac{1}{3}\mathrm{Li}_2(-x)+\frac{1}{2}Y^2-\frac{13}{6}Y+\frac{193}{54}-\frac{\pi^2}{18}\right]\frac{x}{y}+\left[\frac{8}{9}-Y\right]\frac{1}{y}-\\
&\qquad -\left[\frac{8}{9}Y-\frac{1}{3}Y^2\right]\frac{1}{xy}\Bigg\}, \qquad\qquad (A.40)
\end{aligned}
$$

$$
\begin{aligned}
E_4^{[1]} =& \left[-\frac{2}{3}\mathrm{Li}_3(-x)-\frac{4}{3}\mathrm{Li}_3(-y)-\frac{1}{3}(4Y-2X)\mathrm{Li}_2(-x)+\frac{2}{9}X^3+\left(\frac{1}{3}Y-\frac{29}{18}\right)X^2+\right.\\
&+\left(\frac{13}{36}\pi^2-\frac{5}{3}Y^2+\frac{62}{27}+\frac{11}{9}Y\right)X+\left(\frac{107}{72}\pi^2+\frac{11}{27}\right)Y-\frac{685}{162}-\frac{11}{72}\pi^2-\\
&\left.-\frac{11}{36}\zeta_3\right]\frac{x}{y}-\frac{1}{9}(4\pi^2+16Y)\frac{1}{y}-\frac{2}{9}Y^3-\frac{1}{9}Y^2-\frac{1}{9}(8Y^2-2Y^3)\frac{1}{x}+\frac{4}{9}\frac{\pi^2}{xy}Y+\\
&+i\pi\Bigg\{\left[-\frac{2}{3}\mathrm{Li}_2(-x)+\frac{2}{3}X^2-\left(2+\frac{4}{3}Y\right)X+\frac{53}{72}\pi^2+\frac{73}{27}-\frac{1}{3}Y^2+\frac{13}{9}Y\right]\frac{x}{y}+\\
&\qquad+\left[2Y-\frac{16}{9}\right]\frac{1}{y}+\left[\frac{16}{9}Y-\frac{2}{3}Y^2\right]\frac{1}{xy}\Bigg\}, \qquad\qquad (A.41)
\end{aligned}
$$

$$
F_4^{[1]} = \left[\frac{1}{9}Y^2-\frac{10}{27}Y-\frac{\pi^2}{9}+\frac{25}{81}\right]\frac{x}{y}+i\pi\left[\frac{2}{9}Y-\frac{10}{27}\right]\frac{x}{y}, \qquad\qquad (A.42)
$$

For $h=4$ in eq. (2.14) and color factor $\mathrm{Tr}^{[2]}$ in eq. (2.20):

$$
\begin{aligned}
G_4^{[2]} =& \left[-3\mathrm{Li}_4\left(-\frac{x}{y}\right)+2\mathrm{Li}_4(-x)-\mathrm{Li}_4(-y)-\left(\frac{28}{3}-2Y+3X\right)\mathrm{Li}_3(-x)-\frac{1}{24}X^4-\right.\\
&-\frac{7}{24}Y^4+\left(2Y-\frac{25}{3}-X\right)\mathrm{Li}_3(-y)-\left(\frac{25}{3}Y-\frac{\pi^2}{2}-\frac{28}{3}X-\frac{1}{2}X^2\right)\mathrm{Li}_2(-x)+\\
&+\left(\frac{11}{18}+\frac{1}{12}Y\right)X^3-\left(\frac{7}{8}Y^2-\frac{47}{12}Y+\frac{3}{2}x+\frac{385}{72}-\frac{\pi^2}{24}\right)X^2+\frac{17}{12}Y^3X-\\
&-\frac{205}{24}Y^2X+\left(\frac{545}{72}+\frac{5}{6}\pi^2+3x\right)YX+\left(\frac{79}{27}+\frac{3}{2}\zeta_3+\frac{263}{144}\pi^2\right)X+\frac{113}{72}Y^3-\\
&-\left(\frac{67}{12}+\frac{3}{2}x+\frac{5}{6}\pi^2\right)Y^2+\left(2\zeta_3+\frac{1513}{432}+\frac{29}{6}\pi^2\right)Y-\frac{23213}{5184}-\frac{193}{1440}\pi^4-\\
&\left.-\frac{65}{72}\zeta_3-\frac{14}{9}\pi^2-\frac{3}{2}\pi^2x\right]\frac{x}{y}+\\
&+\left[6\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)-\mathrm{Li}_3(-x)\right)-(4+6Y-6X)\mathrm{Li}_3(-y)-\right.\\
&\quad-(4Y+\pi^2-6X)\mathrm{Li}_2(-x)+\left(3Y+\frac{3}{2}Y^2\right)X^2-Y^3X-\frac{19}{4}Y^2X+\\
&\quad+\left(\frac{7}{4}-2\pi^2\right)YX+\left(\frac{5}{6}\pi^2-6\zeta_3\right)X+\frac{13}{12}Y^3-\left(\frac{11}{4}+\frac{\pi^2}{2}\right)Y^2+\\
&\quad\left.+\left(\frac{29}{12}\pi^2+6\zeta_3\right)Y-\frac{17}{12}\pi^2-\zeta_3-\frac{\pi^4}{15}\right]\frac{1}{y}+\\
&+\left[8\mathrm{Li}_4\left(-\frac{x}{y}\right)+8\mathrm{Li}_4(-x)+5\mathrm{Li}_4(-y)-2Y\mathrm{Li}_3(-x)+2X^2Y^2+\frac{1}{8}Y^4+\frac{7}{24}Y^3+\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(7X-\frac{1}{2}-7Y\right)\mathrm{Li}_3(-y)+\left(YX-\frac{4}{3}\pi^2-\frac{1}{2}Y^2-\frac{1}{2}Y\right)\mathrm{Li}_2(-x)-\\
&-\left(\frac{5}{8}Y^2+\frac{5}{3}Y^3+7\zeta_3+2\pi^2Y\right)X-\left(\frac{11}{24}\pi^2+\frac{5}{4}\right)Y^2+\left(7\zeta_3+\frac{\pi^2}{6}\right)Y+\\
&+\frac{1}{2}\zeta_3-\frac{\pi^4}{18}\bigg]\frac{1}{xy}+\\
&+i\pi\Bigg\{\bigg[-\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)+(1+X)\mathrm{Li}_2(-x)+\frac{1}{12}X^3+\left(\frac{13}{12}-Y\right)X^2+\\
&\quad+\left(\frac{5}{12}\pi^2+\frac{3}{2}Y^2-\frac{25}{8}-\frac{11}{12}Y\right)X-\frac{1}{12}Y^3+\frac{1}{3}Y^2-\left(\frac{\pi^2}{2}+\frac{259}{72}\right)Y+\\
&\quad+\frac{2777}{432}+\frac{7}{2}\zeta_3+\frac{37}{48}\pi^2\bigg]\frac{x}{y}+\\
&\quad+\left[2\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y+\frac{7}{4}\right)X-\frac{\pi^2}{12}+\frac{1}{2}Y^2-\frac{15}{4}Y\right]\frac{1}{y}+\\
&\quad+\bigg[-2\mathrm{Li}_3(-x)+\left(X-\frac{1}{2}\right)\mathrm{Li}_2(-x)-\frac{3}{4}YX+\frac{1}{6}Y^3+\frac{1}{2}Y^2-\\
&\quad\quad-\left(\frac{5}{2}-\frac{\pi^2}{12}\right)Y\bigg]\frac{1}{xy}\Bigg\},
\end{aligned}
\tag{A.43}
$$

$$
\begin{aligned}
H_4^{[2]}=&\bigg[\frac{9}{2}\mathrm{Li}_4(-y)-\left(2X+\frac{1}{3}-Y\right)\mathrm{Li}_3(-x)-\left(3X+\frac{8}{3}+Y\right)\mathrm{Li}_3(-y)+\frac{5}{6}Y^3X-\\
&-\frac{1}{8}X^4+\left(2X^2-\left(4Y-\frac{1}{3}\right)X-\frac{8}{3}Y+\frac{5}{4}Y^2\right)\mathrm{Li}_2(-x)+\left(\frac{103}{36}+\frac{17}{12}Y\right)X^3+\\
&+\left(\frac{\pi^2}{8}-\frac{193}{18}-\frac{61}{12}Y-2x-\frac{25}{8}Y^2\right)X^2-\frac{1}{3}Y^2X+\left(\frac{5}{6}\pi^2+4x+\frac{797}{72}\right)YX+\\
&+\left(\frac{241}{27}+\frac{395}{144}\pi^2+\frac{7}{2}\zeta_3\right)X-\left(\frac{\pi^2}{6}+\frac{41}{18}+2x\right)Y^2+\\
&+\left(\frac{235}{54}-\frac{3}{2}\zeta_3+\frac{365}{144}\pi^2\right)Y-\frac{97}{48}\pi^2-\frac{107}{288}\pi^4-\frac{30659}{1296}+\frac{719}{72}\zeta_3-2\pi^2x\bigg]\frac{x}{y}+\\
&+\bigg[-\frac{5}{2}(\mathrm{Li}_3(-y)+Y\mathrm{Li}_2(-x))-\left(2Y+\frac{5}{4}Y^2\right)X+\frac{21}{4}Y^2-\left(\frac{32}{9}-\frac{3}{4}\pi^2\right)Y-\\
&-\frac{11}{36}\pi^2+4\zeta_3\bigg]\frac{1}{y}-\frac{13}{36}Y^3+\frac{25}{36x}Y^3+\\
&+\bigg[-4\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)-Y(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))\right)-\frac{3}{2}\mathrm{Li}_4(-y)-\\
&-\left(2X+\frac{1}{2}\right)\mathrm{Li}_3(-y)-\left(\frac{1}{2}Y+2YX-\pi^2-\frac{5}{4}Y^2\right)\mathrm{Li}_2(-x)-X^2Y^2-\\
&-\frac{1}{6}Y^4+\left(\frac{19}{12}Y^3-\frac{1}{4}Y^2+\frac{\pi^2}{3}Y+2\zeta_3\right)X+\left(\frac{127}{36}-\frac{5}{24}\pi^2\right)Y^2+\\
&+\left(\frac{25}{18}\pi^2-4\zeta_3\right)Y+\frac{1}{2}\zeta_3+\frac{\pi^4}{60}\bigg]\frac{1}{xy}+
\end{aligned}
$$

$$
\begin{aligned}
&+i\pi\Bigg\{\Bigg[-\mathrm{Li}_3(-x)-4\mathrm{Li}_3(-y)-\left(\frac{7}{3}+\frac{3}{2}Y\right)\mathrm{Li}_2(-x)-\frac{1}{12}X^3+\frac{7}{12}Y^3+\\
&\qquad+\left(\frac{10}{3}+\frac{3}{2}Y\right)X^2-\left(\frac{49}{6}Y+\frac{7}{2}Y^2+\frac{83}{8}-\frac{\pi^2}{4}\right)X+\frac{25}{12}Y^2+\\
&\qquad+\left(\frac{469}{72}+\frac{4}{3}\pi^2\right)Y+2\zeta_3+\frac{29}{9}\pi^2+\frac{239}{18}\Bigg]\frac{x}{y}+\\
&\quad+\left[\frac{17}{2}Y-Y^2+\frac{\pi^2}{12}-\frac{32}{9}-2X-\frac{5}{2}\mathrm{Li}_2(-x)\right]\frac{1}{y}+\\
&\quad+\Bigg[4\mathrm{Li}_3(-x)+2\mathrm{Li}_3(-y)-\left(\frac{1}{2}-\frac{1}{2}Y+2X\right)\mathrm{Li}_2(-x)+\frac{1}{2}Y^2X-\frac{1}{6}Y^3-\\
&\qquad-\frac{25}{12}Y^2-\left(\frac{5}{12}\pi^2-\frac{127}{18}\right)Y-2\zeta_3\Bigg]\frac{1}{xy}\Bigg\}\,,
\end{aligned}
\tag{A.44}
$$

$$
\begin{aligned}
I_4^{[2]}=&\Bigg[-\mathrm{Li}_4\left(-\frac{x}{y}\right)+2\mathrm{Li}_4(-x)+\frac{7}{2}\mathrm{Li}_4(-y)+\left(\frac{3}{4}+\frac{1}{2}Y\right)X^3+(Y-3X)\mathrm{Li}_3(-x)-\\
&-\left(Y+2X-\frac{1}{2}\right)\mathrm{Li}_3(-y)+\left(\frac{3}{2}X^2+\frac{\pi^2}{6}+\frac{1}{2}Y+\frac{1}{4}Y^2-2YX\right)\mathrm{Li}_2(-x)+\\
&+\left(-\frac{3}{2}Y^2+\frac{\pi^2}{3}-\frac{29}{8}-\frac{3}{2}Y-\frac{1}{2}x\right)X^2+\frac{1}{4}Y^3X+\left(x+\frac{19}{4}+\frac{2}{3}\pi^2\right)YX+\\
&+\left(6+\frac{7}{12}\pi^2+2\zeta_3\right)X-\left(\frac{1}{2}x+\frac{1}{2}+\frac{\pi^2}{6}\right)Y^2-\left(3\zeta_3+\frac{51}{16}-\frac{\pi^2}{2}\right)Y-\\
&-\frac{23}{72}\pi^4-\frac{\pi^2}{2}x+\frac{23}{4}\zeta_3-\frac{511}{64}-\frac{107}{48}\pi^2\Bigg]\frac{x}{y}+\\
&+\Bigg[-\frac{1}{2}\mathrm{Li}_3(-y)-\frac{1}{2}Y\mathrm{Li}_2(-x)-\left(\frac{1}{4}Y+\frac{\pi^2}{6}\right)X+2Y^2-\left(\frac{\pi^2}{6}+3\right)Y+\\
&\quad+3\zeta_3-\frac{\pi^2}{4}\Bigg]\frac{1}{y}-\frac{7}{8}Y^2X+\left(\frac{1}{6}Y^3+\frac{7}{8}Y^2X\right)\frac{1}{x}+\\
&+\Bigg[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)-\frac{5}{2}\mathrm{Li}_4(-y)+2Y\mathrm{Li}_3(-x)+(3Y-1-X)\mathrm{Li}_3(-y)-\\
&\quad-\left(YX-\frac{2}{3}\pi^2-\frac{3}{4}Y^2+Y\right)\mathrm{Li}_2(-x)-\frac{1}{2}Y^2X^2+\left(\zeta_3+\frac{\pi^2}{3}Y+\frac{11}{12}Y^3\right)X-\\
&\quad-\frac{1}{8}Y^4+\left(\frac{\pi^2}{4}+2\right)Y^2+\left(\frac{13}{12}\pi^2-3\zeta_3\right)Y+\frac{\pi^4}{36}+\zeta_3\Bigg]\frac{1}{xy}+\\
&+i\pi\Bigg\{\Bigg[-2\mathrm{Li}_3(-x)-3\mathrm{Li}_3(-y)-\left(\frac{3}{2}Y-\frac{1}{2}-X\right)\mathrm{Li}_2(-x)+\left(\frac{3}{4}+\frac{1}{2}Y\right)X^2+\\
&\qquad+\left(\frac{2}{3}\pi^2-\frac{5}{2}-2Y^2-\frac{7}{4}Y\right)X+\left(\frac{5}{6}\pi^2+\frac{15}{4}\right)Y+\frac{45}{16}-\zeta_3+\frac{5}{6}\pi^2\Bigg]\frac{x}{y}+\\
&\quad+\left[-\frac{1}{2}\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y-\frac{1}{4}\right)X+\frac{15}{4}Y-\frac{\pi^2}{6}-3\right]\frac{1}{y}-\frac{1}{3}Y^3-\frac{5}{8}Y^2+\\
&\quad+\Bigg[2\mathrm{Li}_3(-x)+2\mathrm{Li}_3(-y)+\left(\frac{1}{2}Y-1-X\right)\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y^2-\frac{3}{4}Y\right)X-
\end{aligned}
$$

$$-\left(\frac{\pi^2}{6}-4\right)Y-2\zeta_3\Bigg]\frac{1}{xy}+\left(\frac{1}{3}Y^3+\frac{7}{8}Y^2\right)\frac{1}{x}\Bigg\}, \tag{A.45}$$

$$\begin{aligned}J_4^{[2]} = &\left[\frac{1}{3}\mathrm{Li}_3(-x)+\frac{1}{3}\mathrm{Li}_3(-y)-\frac{1}{3}(X-Y)\mathrm{Li}_2(-x)-\frac{1}{9}X^3-\left(\frac{1}{6}Y-\frac{29}{36}\right)X^2-\right.\\ &-\left(\frac{31}{27}-\frac{2}{3}Y^2+\frac{13}{72}\pi^2+\frac{11}{18}Y\right)X-\frac{5}{18}Y^3+\frac{37}{36}Y^2-\left(\frac{145}{54}+\frac{\pi^2}{3}\right)Y+\\ &\left.+\frac{455}{108}+\frac{37}{36}\zeta_3-\frac{55}{72}\pi^2\right]\frac{x}{y}+\\ &+i\pi\left\{-\frac{1}{3}X^2+\left(\frac{2}{3}Y+1\right)X+\frac{13}{9}Y-\frac{23}{6}-\frac{1}{3}Y^2-\frac{7}{24}\pi^2\right\}\frac{x}{y},\end{aligned} \tag{A.46}$$

$$\begin{aligned}K_4^{[2]} = &\left[\frac{1}{3}\mathrm{Li}_3(-x)+\frac{2}{3}\mathrm{Li}_3(-y)+\frac{1}{3}(2Y-X)\mathrm{Li}_2(-x)-\frac{1}{9}X^3-\left(\frac{1}{6}Y-\frac{29}{36}\right)X^2-\right.\\ &\left.-\left(\frac{13}{72}\pi^2-\frac{5}{6}Y^2+\frac{31}{27}+\frac{11}{18}Y\right)X-\left(\frac{55}{72}\pi^2+\frac{35}{27}\right)Y+\frac{685}{162}+\frac{23}{36}\zeta_3+\frac{\pi^2}{8}\right]\frac{x}{y}+\\ &+\frac{1}{9}(8Y+2\pi^2)\frac{1}{y}+\frac{1}{18}Y^2+\frac{1}{9}Y^3+\frac{1}{9}(4Y^2-Y^3)\frac{1}{x}-\frac{2}{9}\frac{\pi^2}{xy}Y+\\ &+i\pi\left\{\left[\frac{1}{3}\mathrm{Li}_2(-x)-\frac{1}{3}X^2+\left(\frac{2}{3}Y+1\right)X-\frac{7}{18}\pi^2+\frac{1}{6}Y^2-\frac{13}{18}Y-\frac{22}{9}\right]\frac{x}{y}-\right.\\ &\left.-\left[Y-\frac{8}{9}\right]\frac{1}{y}-\left[\frac{8}{9}Y-\frac{1}{3}Y^2\right]\frac{1}{xy}\right\},\end{aligned} \tag{A.47}$$

$$L_4^{[2]} = -\left[\frac{1}{9}Y^2-\frac{10}{27}Y-\frac{\pi^2}{9}+\frac{25}{81}\right]\frac{x}{y}-i\pi\left[\frac{2}{9}Y-\frac{10}{27}\right]\frac{x}{y}, \tag{A.48}$$

For $h=5$ in eq. (2.15) and color factor $\mathrm{Tr}^{[1]}$ in eq. (2.21):

$$\begin{aligned}A_5^{[1]} = &\left[\frac{3}{2}x\mathrm{Li}_4\left(-\frac{x}{y}\right)-\left(x(Y-X)+\frac{1}{2}\right)(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))+\frac{1}{16}xX^4+\frac{5}{48}xY^4-\right.\\ &-\left(\frac{1}{4}xX^2-\frac{1}{2}(xY+1)X+\frac{\pi^2}{4}x+\frac{1}{4}xY^2+\frac{1}{2}Y\right)\mathrm{Li}_2(-x)-\frac{2}{3}xY^3X-\\ &-\left(\frac{1}{4}xY+\frac{37}{72}x-\frac{1}{3}\right)X^3+\left(\frac{3}{4}xY^2+\left(\frac{5}{8}x-\frac{3}{4}\right)Y+\frac{1}{6}+\left(\frac{\pi^2}{8}+\frac{16}{9}\right)x\right)X^2+\\ &+\left(\frac{1}{2}+\frac{7}{24}x\right)Y^2X-\left(\frac{13}{6}+\left(\frac{32}{9}+\frac{5}{12}\pi^2\right)x\right)YX+\left(\frac{32}{9}+\frac{\pi^2}{3}-\frac{37}{72}\pi^2x\right)X-\\ &-\left(\frac{1}{4}+\frac{29}{72}x\right)Y^3+\left(2+\left(\frac{\pi^2}{6}+\frac{16}{9}\right)x\right)Y^2-\left(\frac{29}{72}\pi^2x+\frac{32}{9}\right)Y+\\ &\left.+\left(\frac{11}{240}\pi^4+\frac{16}{9}\pi^2\right)x+\frac{3}{2}\zeta_3+\frac{43}{36}\pi^2\right]\frac{x}{y}+\\ &+\left[-\frac{1}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)+\frac{4}{3}(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))+\frac{1}{48}X^4-\left(\frac{37}{72}+\frac{1}{4}Y\right)X^3+\right.\\ &\quad+\frac{1}{16}Y^4-\frac{1}{3}Y^3X-\left(\frac{1}{4}X^2+\left(\frac{4}{3}-\frac{1}{2}Y\right)X-\frac{4}{3}Y-\frac{\pi^2}{12}+\frac{1}{4}Y^2\right)\mathrm{Li}_2(-x)+\end{aligned}$$

$$+\left(\frac{1}{2}Y^2-\frac{\pi^2}{24}+\frac{28}{9}-\frac{1}{24}Y\right)X^2+\frac{13}{8}Y^2X-\left(\frac{\pi^2}{12}+\frac{23}{9}\right)YX-\frac{5}{8}Y^3-$$
$$-\left(\frac{3}{2}\zeta_3+\frac{31}{48}\pi^2-\frac{17}{27}\right)X+\left(\frac{101}{36}+\frac{\pi^2}{6}\right)Y^2-\left(\frac{3049}{432}+2\zeta_3+\frac{67}{72}\pi^2\right)Y+$$
$$+\frac{173}{72}\zeta_3+\frac{23213}{5184}-\frac{35}{18}\pi^2+\frac{47}{1440}\pi^4\Bigg]\frac{1}{y}+$$
$$+i\pi\Bigg\{\left[-\frac{11}{12}xX^2+\frac{11}{6}(xY-1)X+\frac{11}{6}Y-\frac{11}{12}\pi^2x-\frac{11}{12}xY^2\right]\frac{x}{y}+$$
$$+\Bigg[-\frac{11}{12}X^2+\left(\frac{11}{3}+\frac{11}{6}Y\right)X-\frac{7}{2}\zeta_3+\frac{55}{18}Y-\frac{11}{16}\pi^2-\frac{2777}{432}-$$
$$-\frac{11}{12}Y^2\Bigg]\frac{1}{y}\Bigg\}, \tag{A.49}$$

$$B_5^{[1]}=\Bigg[\left(6+\frac{11}{2}x\right)\mathrm{Li}_4\left(-\frac{x}{y}\right)+(8x+6)(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y))-\frac{3}{16}xX^4-$$
$$-\left(2xX+(5x+6)Y-\frac{3}{2}-x\right)\mathrm{Li}_3(-x)-\left((7x+6)Y+\frac{9}{2}-x\right)\mathrm{Li}_3(-y)+$$
$$+\left(\frac{3}{4}xX^2-\left(\frac{1}{2}xY+\frac{3}{2}+x\right)X-\frac{1}{4}xY^2-\left(\frac{9}{2}-x\right)Y-\frac{11}{12}\pi^2x-\pi^2\right)\mathrm{Li}_2(-x)-$$
$$-\left(\frac{5}{6}-\frac{11}{12}xY-\frac{101}{72}x\right)X^3+\left(-\frac{5}{4}xY^2-\frac{1}{2}(5x-3)Y+\frac{17}{12}-\frac{14}{9}x\right)X^2-$$
$$-\left(\frac{3}{2}Y^3+\left(\frac{7}{2}-\frac{19}{24}x\right)Y^2-\left(\frac{7}{12}+\left(\frac{28}{9}+\frac{\pi^2}{2}\right)x\right)Y+\frac{37}{9}+\frac{3}{4}\pi^2-\frac{89}{72}\pi^2x\right)X+$$
$$+\left(\frac{1}{2}+\frac{17}{48}x\right)Y^4-\left(2+\left(\frac{14}{9}+\frac{5}{24}\pi^2\right)x\right)Y^2+\left(\frac{5}{3}\pi^2+\frac{11}{36}\pi^2x+\frac{37}{9}\right)Y-$$
$$-\left(\frac{61}{144}\pi^4+\frac{14}{9}\pi^2\right)x+\frac{11}{2}\zeta_3+\frac{\pi^2}{9}-\frac{\pi^4}{4}\Bigg]\frac{x}{y}+$$
$$+\Bigg[-\frac{5}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)-2\mathrm{Li}_4(-y)+\left(2Y-\frac{19}{6}-X\right)\mathrm{Li}_3(-x)+\frac{1}{48}Y^4-$$
$$-\left(\frac{77}{6}-4Y+3X\right)\mathrm{Li}_3(-y)-\frac{1}{16}X^4-\left(\frac{1}{6}Y+\frac{403}{72}+\frac{9}{4}Y^2\right)X^2+Y^3X+$$
$$+\left(\frac{3}{4}X^2+\left(\frac{19}{6}-\frac{5}{2}Y\right)X-\frac{77}{6}Y+\frac{7}{4}Y^2+\frac{5}{12}\pi^2\right)\mathrm{Li}_2(-x)-\frac{215}{24}Y^2X+$$
$$+\left(\frac{89}{72}+\frac{7}{12}Y\right)X^3+\left(\frac{7}{6}\pi^2+\frac{49}{36}\right)YX-\left(\frac{115}{27}-\frac{7}{4}\pi^2-4\zeta_3\right)X-$$
$$-\left(\frac{31}{24}\pi^2-\frac{73}{18}\right)Y^2+\left(\frac{421}{48}\pi^2-\frac{5}{2}\zeta_3-\frac{205}{54}\right)Y+\frac{217}{72}\zeta_3+\frac{859}{144}\pi^2+\frac{30659}{1296}-$$
$$-\frac{271}{1440}\pi^4\Bigg]\frac{1}{y}-\frac{\pi^2}{4}(1-x)X^2-\left(1-\frac{3}{2}x\right)Y^3X-\frac{1}{36}(23x-47)Y^3+\frac{5}{2xy}Y^2+$$
$$+i\pi\Bigg\{\Bigg[-(7x+6)(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))+(x(X-Y)-6)\mathrm{Li}_2(-x)+\frac{53}{24}\pi^2x+$$

$$+\left(\frac{1}{2}xY+\frac{53}{24}x-\frac{1}{4}\right)X^2+\left(Y^2-\left(\frac{53}{12}x-\frac{1}{2}\right)Y+\frac{\pi^2}{6}x+\frac{41}{12}\right)X+$$
$$+\left(\frac{7}{6}x+1\right)Y^3+\left(\frac{53}{24}x-\frac{13}{4}\right)Y^2+\left(\frac{3}{2}\pi^2x+\pi^2-\frac{41}{12}\right)Y-\frac{\pi^2}{12}\Bigg]\frac{x}{y}+$$
$$+\Bigg[\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)-\left(X-Y+\frac{29}{3}\right)\mathrm{Li}_2(-x)+\frac{47}{24}X^2+\frac{5}{6}Y^3-$$
$$-\frac{155}{24}Y^2-\left(\frac{59}{6}+\frac{65}{12}Y-\frac{\pi^2}{2}\right)X+\left(\frac{341}{36}-\frac{\pi^2}{3}\right)Y+\frac{3}{2}\zeta_3-$$
$$-\frac{145}{18}+\frac{259}{144}\pi^2\Bigg]\frac{1}{y}+4xY^2X+\frac{5}{xy}Y\Bigg\}, \tag{A.50}$$

$$C_5^{[1]}=\Bigg[-6x\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)\right)+(2xY+xX-3x-2)\mathrm{Li}_3(-x)+$$
$$+(8xY-3x-5xX-2)\mathrm{Li}_3(-y)-\frac{1}{6}xY^4-\left(\frac{9}{2}+\left(\frac{7}{24}\pi^2+\frac{9}{2}\right)x\right)Y^2+$$
$$+\frac{1}{6}xX^4-\left(xX^2+(xY-2-3x)X-2xY^2+(2+3x)Y-\pi^2x\right)\mathrm{Li}_2(-x)-$$
$$-\left(\frac{23}{24}x-\frac{3}{4}+\frac{7}{6}xY\right)X^3-\left(\frac{1}{2}xY^2-\left(\frac{13}{4}x-\frac{1}{2}\right)Y+\frac{15}{4}+\left(\frac{9}{2}-\frac{\pi^2}{24}\right)x\right)X^2+$$
$$+\left(\frac{13}{6}xY^3+\left(\frac{33}{4}+\left(9+\frac{3}{4}\pi^2\right)x\right)Y-\frac{11}{24}\pi^2x+\frac{2}{3}\pi^2-6\right)X+$$
$$+\left(6+\frac{4}{3}\pi^2x+\frac{11}{12}\pi^2\right)Y+\left(-\frac{9}{2}\pi^2+\frac{7}{24}\pi^4\right)x-6\zeta_3-\frac{10}{3}\pi^2\Bigg]\frac{x}{y}+$$
$$+\Bigg[10\mathrm{Li}_4\left(-\frac{x}{y}\right)+6\mathrm{Li}_4(-x)-2\mathrm{Li}_4(-y)+(1+3X-6Y)\mathrm{Li}_3(-x)+\frac{1}{24}X^4+$$
$$+(1-4Y+9X)\mathrm{Li}_3(-y)+\left(-X^2-(1-5Y)X+Y-\frac{13}{3}\pi^2\right)\mathrm{Li}_2(-x)-$$
$$-\frac{3}{2}Y^3X+\frac{1}{3}Y^4-\left(\frac{1}{3}Y+\frac{13}{24}\right)X^3-\left(\frac{5}{8}\pi^2-\frac{3}{4}Y-\frac{1}{8}-4Y^2\right)X^2+$$
$$+\left(\frac{29}{4}-\frac{35}{12}\pi^2\right)YX-\left(4\zeta_3-6+\frac{9}{8}\pi^2\right)X-\left(\frac{29}{24}\pi^2-\frac{3}{2}\right)Y^2+$$
$$+\left(\frac{99}{16}+7\zeta_3-\frac{31}{6}\pi^2\right)Y-\frac{39}{4}\zeta_3+\frac{511}{64}-\frac{269}{48}\pi^2+\frac{247}{360}\pi^4\Bigg]\frac{1}{y}-$$
$$-\frac{1}{8}(19-29x)Y^2X-\frac{1}{3}xY^3+\frac{3}{2xy}Y^2+$$
$$+i\pi\Bigg\{\Bigg[3x(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)-(X-Y)\mathrm{Li}_2(-x))-\left(\frac{3}{2}xY+\frac{9}{8}x-\frac{3}{4}\right)X^2+\frac{\pi^2}{4}+$$
$$+\left(3xY^2-\left(\frac{3}{2}-\frac{9}{4}x\right)Y-\frac{\pi^2}{2}x+\frac{3}{4}\right)X-\left(\frac{3}{2}\pi^2x+\frac{3}{4}\right)Y-\frac{9}{8}\pi^2x\Bigg]\frac{x}{y}+$$
$$+\Bigg[-3\mathrm{Li}_3(-x)+5\mathrm{Li}_3(-y)+(3X+5Y)\mathrm{Li}_2(-x)-\frac{3}{8}X^2+4Y^2X+$$

$$+\frac{21}{4}YX+\left(\frac{15}{2}-\frac{3}{2}\pi^2\right)X-\left(\frac{4}{3}\pi^2-\frac{41}{4}\right)Y+3\zeta_3+\frac{195}{16}-\frac{9}{8}\pi^2\Bigg]\frac{1}{y}-$$

$$-\frac{1}{2}(1-x)Y^3+\frac{1}{8}(9x-15)Y^2+\frac{3}{xy}Y\Bigg\}, \tag{A.51}$$

$$D_5^{[1]} = \left[\frac{1}{18}xX^3-\left(\frac{4}{9}x+\frac{1}{6}\right)X^2-\left(\frac{1}{6}xY^2-\left(\frac{8}{9}x+\frac{2}{3}\right)Y+\frac{8}{9}-\frac{\pi^2}{18}x\right)X+\frac{1}{9}xY^3-\right.$$

$$\left.-\left(\frac{4}{9}x+\frac{1}{2}\right)Y^2+\frac{1}{9}(\pi^2x+8)Y-\frac{5}{18}\pi^2-\frac{4}{9}\pi^2x\right]\frac{x}{y}+$$

$$+\left[-\frac{1}{3}\mathrm{Li}_3(-x)-\frac{1}{3}\mathrm{Li}_3(-y)+\frac{1}{3}(X-Y)\mathrm{Li}_2(-x)+\frac{1}{18}X^3+\left(\frac{1}{6}Y-\frac{19}{36}\right)X^2+\right.$$

$$+\left(\frac{7}{27}+\frac{\pi^2}{8}+\frac{7}{18}Y-\frac{1}{2}Y^2\right)X+\frac{1}{6}Y^3-\frac{13}{12}Y^2+\left(\frac{193}{54}+\frac{2}{9}\pi^2\right)Y+$$

$$\left.+\frac{67}{72}\pi^2-\frac{455}{108}-\frac{37}{36}\zeta_3\right]\frac{1}{y}+$$

$$+i\pi\left\{\left[\frac{1}{6}xX^2+\frac{1}{3}(1-xY)X-\frac{1}{3}Y+\frac{1}{6}xY^2+\frac{\pi^2}{6}x\right]\frac{x}{y}+\right.$$

$$\left.+\left[\frac{1}{6}X^2-\frac{1}{3}(2+Y)X-\frac{16}{9}Y+\frac{\pi^2}{8}+\frac{1}{6}Y^2+\frac{23}{6}\right]\frac{1}{y}\right\}, \tag{A.52}$$

$$E_5^{[1]} = \left[-\frac{1}{9}xX^3+\left(\frac{8}{9}x+\frac{1}{3}\right)X^2+\left(\frac{1}{3}xY^2-\left(\frac{4}{3}+\frac{16}{9}x\right)Y+\frac{16}{9}-\frac{\pi^2}{9}x\right)X-\right.$$

$$\left.-\frac{1}{9}(2\pi^2x+16)Y+\frac{5}{9}\pi^2+\frac{8}{9}\pi^2x\right]\frac{x}{y}+$$

$$+\left[\frac{2}{3}\mathrm{Li}_3(-x)+\frac{4}{3}\mathrm{Li}_3(-y)-\frac{2}{3}(X-2Y)\mathrm{Li}_2(-x)-\frac{1}{9}X^3+\left(\frac{19}{18}-\frac{1}{3}Y\right)X^2-\right.$$

$$-\left(\frac{\pi^2}{4}+\frac{7}{9}Y-\frac{4}{3}Y^2+\frac{14}{27}\right)X-\left(\frac{13}{8}\pi^2-\frac{11}{27}\right)Y-\frac{685}{162}-\frac{59}{72}\pi^2-$$

$$\left.-\frac{59}{36}\zeta_3\right]\frac{1}{y}-\frac{2}{9}(1-x)Y^3-\frac{1}{9}(1+8x)Y^2+$$

$$+i\pi\left\{\left[-\frac{1}{3}xX^2-\frac{2}{3}(1-xY)X+\frac{2}{3}Y-\frac{\pi^2}{3}x-\frac{1}{3}xY^2\right]\frac{x}{y}+\right.$$

$$\left.+\left[\frac{2}{3}\mathrm{Li}_2(-x)-\frac{1}{3}X^2+\frac{2}{3}(2+Y)X-\frac{1}{9}-\frac{23}{72}\pi^2-\frac{5}{9}Y+\frac{4}{3}Y^2\right]\frac{1}{y}\right\}, \tag{A.53}$$

$$F_5^{[1]} = \left[\frac{1}{9}Y^2-\frac{10}{27}Y-\frac{\pi^2}{9}+\frac{25}{81}\right]\frac{1}{y}+i\pi\left[\frac{2}{9}Y-\frac{10}{27}\right]\frac{1}{y}, \tag{A.54}$$

For $h=5$ in eq. (2.15) and color factor $\mathrm{Tr}^{[2]}$ in eq. (2.21):

$$G_5^{[2]} = \left[-(5x+6)\mathrm{Li}_4\left(-\frac{x}{y}\right)-(8x+6)(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y))-\left(\frac{5}{12}x+\frac{1}{2}\right)Y^4+\right.$$

$$+\left(2xX+(6+5x)Y+\frac{1}{2}x-2\right)\mathrm{Li}_3(-x)+\left((7x+6)Y+4+\frac{1}{2}x\right)\mathrm{Li}_3(-y)-$$

$$
-\left(\frac{1}{2}xX^2-\left(-\frac{1}{2}x+2\right)X-\frac{1}{2}xY^2-\left(4+\frac{1}{2}x\right)Y-\pi^2-\frac{5}{6}\pi^2x\right)\mathrm{Li}_2(-x)-
$$

$$
-\frac{1}{6}xX^3Y+\left(\frac{1}{4}xY^2+\left(\frac{1}{8}x+\frac{3}{4}\right)Y-1+\frac{\pi^2}{8}-\frac{5}{4}x\right)X^2+
$$

$$
+\left(\frac{3}{2}-\frac{1}{4}x\right)Y^2X+\left(\frac{15}{4}-\left(\frac{\pi^2}{4}-\frac{5}{2}\right)x\right)YX-\frac{\pi^2}{12}(x-5)X+
$$

$$
+\left(\frac{13}{12}+\frac{7}{24}x\right)Y^3-\left(\frac{11}{4}+\left(\frac{5}{4}-\frac{\pi^2}{24}\right)x\right)Y^2-\left(\frac{7}{6}\pi^2-\frac{\pi^2}{8}x\right)Y+
$$

$$
+\left(\frac{17}{40}\pi^4-\frac{5}{4}\pi^2\right)x-\frac{29}{12}\pi^2+\frac{\pi^4}{4}-5\zeta_3\Bigg]\frac{x}{y}-2xY^3X+
$$

$$
+\Bigg[\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)+3\mathrm{Li}_4(-y)+(2X-1)\mathrm{Li}_3(-x)-\frac{1}{8}Y^4+\frac{1}{12}YX^3+
$$

$$
+\left(\frac{25}{3}-2Y+X\right)\mathrm{Li}_3(-y)+\left(\frac{25}{3}Y+X-\frac{\pi^2}{2}-\frac{1}{2}X^2\right)\mathrm{Li}_2(-x)+
$$

$$
+\frac{113}{72}Y^3+\left(\frac{3}{8}Y^2-\frac{\pi^2}{4}-\frac{1}{8}Y\right)X^2-\frac{5}{12}Y^3X+\frac{17}{3}Y^2X+\left(\frac{\pi^2}{4}-\zeta_3\right)X+
$$

$$
+\left(\frac{19}{8}-\frac{11}{12}\pi^2\right)YX-\left(\frac{67}{12}-\frac{3}{4}\pi^2\right)Y^2+\left(\frac{1513}{432}-\frac{407}{72}\pi^2+4\zeta_3\right)Y-
$$

$$
-\frac{41}{24}\pi^2+\frac{223}{1440}\pi^4-\frac{23213}{5184}-\frac{665}{72}\zeta_3\Bigg]\frac{1}{y}-\left(\frac{\pi^2}{8}x-\frac{\pi^2}{4}\right)X^2-\frac{3}{2xy}Y^2+
$$

$$
+i\pi\Bigg\{\Bigg[(7x+6)(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))-(xX-6-xY)\mathrm{Li}_2(-x)-\frac{1}{6}xX^3+
$$

$$
+\left(\left(\left(\frac{1}{2}-\frac{3}{4}x\right)Y-\frac{\pi^2}{3}x+\frac{7}{4}\right)X+\left(\frac{3}{8}x+\frac{11}{4}\right)Y^2-\right.
$$

$$
\left.-\left(\pi^2+\frac{7}{4}+\frac{4}{3}\pi^2x\right)Y+\frac{3}{8}\pi^2x-\frac{\pi^2}{12}\right]\frac{x}{y}-\frac{3}{xy}Y+xY^3+
$$

$$
+\Bigg[2\mathrm{Li}_3(-x)-\mathrm{Li}_3(-y)+\left(\frac{28}{3}-X\right)\mathrm{Li}_2(-x)+\frac{1}{12}X^3+\frac{1}{2}X^2Y-
$$

$$
-\frac{7}{12}Y^3+\left(\frac{19}{8}-\frac{3}{4}\pi^2+\frac{11}{4}Y\right)X+\frac{149}{24}Y^2-\left(\frac{211}{24}-\frac{\pi^2}{4}\right)Y-
$$

$$
-\frac{37}{72}\pi^2+\frac{1513}{432}+3\zeta_3\Bigg]\frac{1}{y}+\frac{1}{8}(5-3x)X^2+\frac{1}{2}(1-7x)Y^2X\Bigg\}, \qquad \text{(A.55)}
$$

$$
H_5^{[2]}=\Bigg[\frac{3}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)x+4x(\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y))+\left(\frac{5}{2}+\frac{1}{2}x-2xX\right)\mathrm{Li}_3(-x)+
$$

$$
+\left(\frac{1}{2}x+2xX+\frac{5}{2}-4xY\right)\mathrm{Li}_3(-y)-\frac{7}{6}xY^3X-\left(\frac{1}{12}-\frac{1}{4}xY+\frac{11}{36}x\right)X^3+
$$

$$
+\left(\frac{3}{4}xX^2-\frac{1}{2}(x+5-xY)X-\frac{5}{4}xY^2+\frac{1}{2}(5+x)Y-\frac{\pi^2}{4}x\right)\mathrm{Li}_2(-x)+
$$

$$
+\frac{1}{48}xX^4+\frac{1}{16}xY^4+\left(\frac{1}{2}xY^2-\left(1+\frac{1}{4}x\right)Y+\frac{17}{12}-\left(\frac{\pi^2}{24}-\frac{127}{36}\right)x\right)X^2+
$$

$$
\begin{aligned}
&+\left(\frac{9}{4}+\frac{17}{12}x\right)Y^2X-\left(\frac{20}{3}+\left(\frac{127}{18}+\frac{\pi^2}{4}\right)x\right)YX+\left(\frac{127}{36}\pi^2-\frac{37}{144}\pi^4\right)x-\\
&-\left(\frac{7}{18}\pi^2x-\frac{32}{9}+\frac{7}{12}\pi^2\right)X+\left(\frac{21}{4}+\left(\frac{\pi^2}{6}+\frac{127}{36}\right)x\right)Y^2-\\
&-\left(\frac{13}{12}\pi^2+\frac{32}{9}+\frac{31}{36}\pi^2x\right)Y+\frac{53}{18}\pi^2+\frac{3}{2}\zeta_3\Big]\frac{x}{y}+\frac{1}{36}(25x-13)Y^3-\frac{2}{xy}Y^2+\\
&+\Big[-\frac{9}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)+\left(\frac{7}{3}-3X+2Y\right)\mathrm{Li}_3(-x)+\left(\frac{8}{3}+Y-4X\right)\mathrm{Li}_3(-y)+\\
&\quad+\left(\frac{3}{4}X^2-\left(\frac{3}{2}Y+\frac{7}{3}\right)X+\frac{8}{3}Y-\frac{5}{4}Y^2+\frac{11}{4}\pi^2\right)\mathrm{Li}_2(-x)+\frac{1}{48}X^4-\\
&\quad-\left(\frac{7}{18}+\frac{1}{6}Y\right)X^3+\left(\frac{3}{4}\pi^2-\frac{1}{6}Y+\frac{125}{72}-\frac{11}{8}Y^2\right)X^2-\frac{1}{12}Y^3X+\frac{1}{6}Y^2X-\\
&\quad-\left(\frac{337}{72}-\frac{7}{4}\pi^2\right)YX+\left(\frac{17}{27}+\frac{3}{2}\zeta_3-\frac{11}{16}\pi^2\right)X-\frac{3}{16}Y^4+\left(\frac{9}{8}\pi^2-\frac{41}{18}\right)Y^2+\\
&\quad+\left(\frac{17}{16}\pi^2-\frac{5}{2}\zeta_3+\frac{235}{54}\right)Y-\frac{661}{1440}\pi^4+\frac{61}{48}\pi^2-\frac{30659}{1296}+\frac{527}{72}\zeta_3\Big]\frac{1}{y}+\\
&+i\pi\Bigg\{\Big[-2x(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)-(X-Y)\mathrm{Li}_2(-x))+\frac{1}{6}xX^3-\frac{11}{12}xX^2+\\
&\qquad+\left(\frac{\pi^2}{2}x-\frac{3}{2}xY^2+\frac{11}{6}xY-\frac{23}{6}\right)X+\frac{1}{6}xY^3-\frac{11}{12}xY^2+\\
&\qquad+\frac{1}{6}(5\pi^2x+23)Y-\frac{11}{12}\pi^2x\Big]\frac{x}{y}+\\
&\qquad+\Big[-\mathrm{Li}_3(-x)-3\mathrm{Li}_3(-y)+\left(\frac{1}{3}-4Y\right)\mathrm{Li}_2(-x)-\frac{1}{12}X^3-\frac{7}{12}Y^3-\frac{1}{6}X^2-\\
&\qquad\quad-\left(2Y^2-\frac{5}{4}\pi^2+\frac{29}{24}+\frac{8}{3}Y\right)X-\frac{1}{12}Y^2-\left(\frac{665}{72}-\frac{11}{12}\pi^2\right)Y-\frac{\pi^2}{72}-\\
&\qquad\quad-\zeta_3+\frac{269}{54}\Big]\frac{1}{y}-\frac{1}{2}(x-1)YX^2-\frac{4}{xy}Y\Bigg\},
\end{aligned}
\tag{A.56}
$$

$$
\begin{aligned}
I_5^{[2]}=&\Big[\frac{5}{2}x\mathrm{Li}_4\left(-\frac{x}{y}\right)+2x\mathrm{Li}_4(-x)+2x\mathrm{Li}_4(-y)-\frac{1}{16}xX^4+\left(x-xY+\frac{1}{2}\right)\mathrm{Li}_3(-x)+\\
&+\left(2Xx-3xY+\frac{1}{2}+x\right)\mathrm{Li}_3(-y)+\left(\frac{5}{12}xY-\frac{1}{6}+\frac{3}{8}x\right)X^3+\frac{1}{16}xY^4+\\
&+\left(\frac{1}{4}xX^2-\left(x+\frac{1}{2}-\frac{1}{2}xY\right)X-\frac{3}{4}xY^2+\left(\frac{1}{2}+x\right)Y-\frac{5}{12}\pi^2x\right)\mathrm{Li}_2(-x)+\\
&+\left(2x-\frac{5}{4}xY+\frac{7}{4}+\frac{1}{4}xY^2\right)X^2-\frac{5}{6}xY^3X-\left(\frac{15}{4}+\left(\frac{\pi^2}{3}+4\right)x\right)YX+\\
&+\left(\frac{5}{24}\pi^2x-\frac{\pi^2}{12}+3\right)X+\left(2+\left(2+\frac{\pi^2}{8}\right)x\right)Y^2-\left(3+\frac{\pi^2}{3}+\frac{\pi^2}{2}x\right)Y+\\
&+\left(2\pi^2-\frac{59}{720}\pi^4\right)x+\frac{3}{2}\pi^2+\frac{5}{2}\zeta_3\Big]\frac{x}{y}-\frac{1}{8}(11x-7)Y^2X+\frac{1}{6}xY^3+
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{7}{2}\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)-\frac{1}{48}X^4-\left(\frac{1}{2}-2Y+X\right)\mathrm{Li}_3(-x)-\right.\\
&\quad-\left(3X+\frac{1}{2}-Y\right)\mathrm{Li}_3(-y)-\frac{5}{48}Y^4+\left(\frac{\pi^2}{6}-\frac{5}{4}Y^2+\frac{5}{8}-\frac{1}{4}Y\right)X^2+\\
&\quad+\left(\frac{1}{4}X^2-\frac{1}{2}(3Y-1)X-\frac{1}{4}Y^2-\frac{1}{2}Y+\frac{19}{12}\pi^2\right)\mathrm{Li}_2(-x)+\left(\frac{1}{12}Y+\frac{5}{24}\right)X^3+\\
&\quad+\left(\frac{1}{3}Y^3+\left(\pi^2-\frac{15}{4}\right)Y+\zeta_3-3+\frac{11}{24}\pi^2\right)X-\left(\frac{1}{2}-\frac{11}{24}\pi^2\right)Y^2-\\
&\quad\left.-\left(4\zeta_3+\frac{51}{16}-\frac{23}{12}\pi^2\right)Y+\frac{25}{4}\zeta_3+\frac{97}{48}\pi^2-\frac{163}{720}\pi^4-\frac{511}{64}\right]\frac{1}{y}-\frac{1}{2xy}Y^2+\\
&+i\pi\left\{\left[-x\mathrm{Li}_3(-x)-x\mathrm{Li}_3(-y)+x(X-Y)\mathrm{Li}_2(-x)+\left(\frac{3}{8}x+\frac{1}{2}xY-\frac{1}{4}\right)X^2-\right.\right.\\
&\qquad\left.-\left(xY^2-\left(\frac{1}{2}-\frac{3}{4}x\right)Y+\frac{1}{4}-\frac{\pi^2}{6}x\right)X+\left(\frac{1}{4}+\frac{\pi^2}{2}x\right)Y-\frac{\pi^2}{12}+\frac{3}{8}\pi^2x\right]\frac{x}{y}+\\
&\qquad+\left[\mathrm{Li}_3(-x)-2\mathrm{Li}_3(-y)-(X+2Y)\mathrm{Li}_2(-x)+\frac{1}{8}X^2-\right.\\
&\qquad\quad\left.-\left(\frac{5}{2}-\frac{\pi^2}{2}+\frac{7}{4}Y+\frac{3}{2}Y^2\right)X-\left(\frac{19}{4}-\frac{\pi^2}{2}\right)Y-\frac{99}{16}-3\zeta_3+\frac{13}{24}\pi^2\right]\frac{1}{y}+\\
&\qquad\left.+\frac{1}{6}(1-x)Y^3-\frac{1}{8}(3x-5)Y^2-\frac{1}{xy}Y\right\}, \qquad \text{(A.57)}
\end{aligned}
$$

$$
\begin{aligned}
J_5^{[2]}=&\left[-\frac{1}{3}(\mathrm{Li}_3(-y)+Y\mathrm{Li}_2(-x))-\frac{1}{6}Y^2X-\frac{5}{18}Y^3+\frac{37}{36}Y^2+\left(\frac{11}{18}\pi^2-\frac{145}{54}\right)Y-\right.\\
&\left.-\frac{41}{72}\pi^2+\frac{49}{36}\zeta_3+\frac{455}{108}\right]\frac{1}{y}+i\frac{\pi}{3y}\left\{\frac{\pi^2}{6}-\frac{5}{2}Y^2+\frac{37}{6}Y-\mathrm{Li}_2(-x)-\frac{145}{18}\right\}, \qquad \text{(A.58)}
\end{aligned}
$$

$$
\begin{aligned}
K_5^{[2]}=&\left[\frac{1}{18}xX^3-\left(\frac{1}{6}+\frac{4}{9}x\right)X^2-\left(\frac{1}{6}xY^2-\left(\frac{2}{3}+\frac{8}{9}x\right)Y-\frac{\pi^2}{18}x+\frac{8}{9}\right)X+\right.\\
&\left.+\frac{1}{9}(8+\pi^2x)Y-\frac{5}{18}\pi^2-\frac{4}{9}\pi^2x\right]\frac{x}{y}+\\
&+\left[-\frac{1}{3}\mathrm{Li}_3(-x)-\frac{2}{3}\mathrm{Li}_3(-y)-\frac{1}{3}(2Y-X)\mathrm{Li}_2(-x)+\frac{1}{18}X^3-\left(\frac{19}{36}-\frac{1}{6}Y\right)X^2+\right.\\
&\quad+\left(\frac{7}{27}+\frac{\pi^2}{8}+\frac{7}{18}Y-\frac{2}{3}Y^2\right)X+\left(\frac{19}{24}\pi^2-\frac{35}{27}\right)Y+\frac{11}{24}\pi^2+\frac{47}{36}\zeta_3+\\
&\quad\left.+\frac{685}{162}\right]\frac{1}{y}+\frac{1}{9}(1-x)Y^3+\left(\frac{1}{18}+\frac{4}{9}x\right)Y^2+\\
&+i\pi\left\{\left[\frac{1}{6}xX^2-\frac{1}{3}(xY-1)X+\frac{\pi^2}{6}x+\frac{1}{6}xY^2-\frac{1}{3}Y\right]\frac{x}{y}-\right.\\
&\qquad\left.-\left[\mathrm{Li}_2(-x)-\frac{1}{2}X^2+(Y+2)X+2Y^2+\frac{28}{9}-\frac{5}{12}\pi^2-\frac{5}{6}Y\right]\frac{1}{3y}\right\}, \qquad \text{(A.59)}
\end{aligned}
$$

$$
L_5^{[2]}=\left[\frac{\pi^2}{9}-\frac{1}{9}Y^2+\frac{10}{27}Y-\frac{25}{81}\right]\frac{1}{y}+i\pi\left[-\frac{2}{9}Y+\frac{10}{27}\right]\frac{1}{y}, \qquad \text{(A.60)}
$$

For $h = 6$ in eq. (2.16) and color factor $\mathrm{Tr}^{[1]}$ in eq. (2.21):

$$
\begin{aligned}
A_6^{[1]} = &\Bigg[\mathrm{Li}_4\left(-\frac{x}{y}\right) - \left(Y - X - \frac{29}{6}\right)(\mathrm{Li}_3(-x) + \mathrm{Li}_3(-y)) + \frac{1}{12}X^4 + \frac{1}{6}Y^4 - Y^3 X - \\
&- \left(\frac{1}{2}X^2 + \left(\frac{29}{6} - Y\right)X + \frac{1}{2}Y^2 + \frac{\pi^2}{6} - \frac{29}{6}Y\right)\mathrm{Li}_2(-x) - \left(\frac{1}{2}Y + \frac{49}{36}\right)X^3 - \\
&- \left(\frac{1}{6}Y - \frac{5}{4}Y^2 - \frac{\pi^2}{12} - \frac{56}{9} - \frac{3}{2}x\right)X^2 + \frac{53}{12}Y^2 X - \left(\frac{125}{18} + \frac{\pi^2}{2} + 3x\right)YX - \\
&- \left(\frac{79}{27} + \frac{287}{144}\pi^2 + \frac{3}{2}\zeta_3\right)X - \frac{23}{18}Y^3 + \left(\frac{3}{2}x + \frac{\pi^2}{3} + \frac{49}{12}\right)Y^2 - \\
&- \left(\frac{17}{6}\pi^2 + \frac{1513}{432} + 2\zeta_3\right)Y + \frac{23}{36}\pi^2 + \frac{23213}{5184} + \frac{113}{1440}\pi^4 + \frac{65}{72}\zeta_3 + \frac{3}{2}\pi^2 x\Bigg]\frac{x}{y} + \\
&+ \Bigg[-3\left(\mathrm{Li}_4\left(-\frac{x}{y}\right) + \mathrm{Li}_4(-x) + \mathrm{Li}_4(-y) - \mathrm{Li}_3(-x) + (X - Y - 1)\mathrm{Li}_3(-y)\right) + \\
&\quad + \left(3Y - 3X + \frac{\pi^2}{2}\right)\mathrm{Li}_2(-x) - \frac{3}{4}(2Y + Y^2)X^2 - \frac{1}{2}Y^3 + \frac{\pi^2}{4}Y^2 + \\
&\quad + \left(3Y^2 + 3\zeta_3 + \pi^2 Y - \frac{\pi^2}{2} + \frac{1}{2}Y^3\right)X - \left(\frac{3}{2}\pi^2 + 3\zeta_3\right)Y + \frac{\pi^4}{30} + \frac{\pi^2}{2}\Bigg]\frac{1}{y} + \\
&+ \Bigg[-3\mathrm{Li}_4\left(-\frac{x}{y}\right) - 3\mathrm{Li}_4(-x) - 3\mathrm{Li}_4(-y) - 3(X - Y)\mathrm{Li}_3(-y) + \frac{\pi^2}{2}\mathrm{Li}_2(-x) - \\
&\quad - \frac{3}{4}Y^2 X^2 + \left(\pi^2 Y + \frac{1}{2}Y^3 + 3\zeta_3\right)X + \frac{\pi^2}{4}Y^2 - 3Y\zeta_3 + \frac{\pi^4}{30}\Bigg]\frac{1}{xy} + \\
&+ i\pi\left\{-\frac{11}{6}(X^2 + Y^2) + \left(\frac{11}{2} + \frac{11}{3}Y\right)X + \frac{11}{9}Y - \frac{7}{2}\zeta_3 - \frac{2777}{432} - \frac{77}{48}\pi^2\right\}\frac{x}{y}, \qquad \text{(A.61)}
\end{aligned}
$$

$$
\begin{aligned}
B_6^{[1]} = &\Bigg[-\mathrm{Li}_4\left(-\frac{x}{y}\right) + 2\mathrm{Li}_4(-x) + 4\mathrm{Li}_4(-y) - \frac{1}{4}X^4 - \frac{1}{8}Y^4 - \left(3X + \frac{29}{3} - Y\right)\mathrm{Li}_3(-x) - \\
&- \left(2X + Y + \frac{37}{3}\right)\mathrm{Li}_3(-y) + \left(\frac{3}{2}Y + \frac{125}{36}\right)X^3 + \left(\frac{4}{3}Y^3 - \frac{229}{24}Y^2\right)X + \\
&+ \left(\frac{3}{2}X^2 - \left(2Y - \frac{29}{3}\right)X - \frac{\pi^2}{6} - \frac{37}{3}Y + \frac{1}{2}Y^2\right)\mathrm{Li}_2(-x) - \\
&- \left(3Y^2 + \frac{7}{6}Y + \frac{7}{2}x + \frac{869}{72}\right)X^2 - \left(\frac{19}{18} + \frac{7}{2}x + \frac{25}{24}\pi^2\right)Y^2 + \\
&+ \left(\left(\frac{4}{3}\pi^2 + 7x + \frac{383}{36}\right)Y - \frac{4}{27} + \frac{329}{72}\pi^2 + 3\zeta_3\right)X - \\
&- \left(\frac{811}{54} - \frac{1267}{144}\pi^2 - \frac{3}{2}\zeta_3\right)Y + \frac{30659}{1296} - \frac{83}{48}\pi^2 + \frac{37}{72}\zeta_3 - \frac{7}{2}\pi^2 x - \frac{13}{32}\pi^4\Bigg]\frac{x}{y} + \\
&+ \Bigg[6\left(\mathrm{Li}_4\left(-\frac{x}{y}\right) + \mathrm{Li}_4(-x) + \mathrm{Li}_4(-y) - \mathrm{Li}_3(-x) + (X - 1 - Y)\mathrm{Li}_3(-y)\right) + \\
&\quad + (6X - \pi^2 - 6Y)\mathrm{Li}_2(-x) + \left(\frac{3}{2}Y^2 + 3Y\right)X^2 - Y^3 X - \frac{23}{4}Y^2 X - \\
&\quad - \left(\frac{1}{4} + 2\pi^2\right)YX - \frac{1}{2}(\pi^2 - 9)Y^2 - \left(\frac{64}{9} - 6\zeta_3 - \frac{15}{4}\pi^2\right)Y +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{5}{6}\pi^2-6\zeta_3\right)X-\frac{91}{36}\pi^2+4\zeta_3-\frac{\pi^4}{15}\bigg]\frac{1}{y}+\frac{1}{36}\left(\frac{29}{x}-47\right)Y^3+\\
&+\bigg[4\mathrm{Li}_4\left(-\frac{x}{y}\right)+4\mathrm{Li}_4(-x)+2\mathrm{Li}_4(-y)+2Y\mathrm{Li}_3(-x)-(2Y+1-5X)\mathrm{Li}_3(-y)-\\
&-\left(Y-Y^2+XY+\frac{\pi^2}{3}\right)\mathrm{Li}_2(-x)+Y^2X^2-\frac{5}{3}\pi^2YX-5\zeta_3X+\frac{1}{6}Y^3X-\\
&-\frac{7}{8}Y^2X+\left(\frac{73}{18}-\frac{23}{24}\pi^2\right)Y^2+\left(2\zeta_3+\frac{85}{36}\pi^2\right)Y-\frac{\pi^4}{45}+\zeta_3\bigg]\frac{1}{xy}+\\
&+i\pi\Bigg\{\bigg[-2\mathrm{Li}_3(-x)-3\mathrm{Li}_3(-y)-\left(\frac{8}{3}+Y-X\right)\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y+\frac{53}{12}\right)X^2-\\
&\quad-\left(\frac{27}{2}+2Y^2+\frac{109}{12}Y-\frac{2}{3}\pi^2\right)X+\frac{5}{6}Y^3+\frac{13}{24}Y^2+\left(\frac{3}{4}\pi^2+\frac{307}{36}\right)Y-\\
&\quad-\frac{91}{6}+\frac{9}{2}\zeta_3+\frac{601}{144}\pi^2\bigg]\frac{x}{y}+\bigg[\left(\frac{1}{2}Y-\frac{1}{4}\right)X-\frac{5}{4}Y^2-\frac{64}{9}+\frac{35}{4}Y+\frac{\pi^2}{12}\bigg]\frac{1}{y}+\\
&\quad+\bigg[2\mathrm{Li}_3(-x)+3\mathrm{Li}_3(-y)-(1+X-Y)\mathrm{Li}_2(-x)+\left(Y^2-\frac{3}{4}Y\right)X+\\
&\quad+\frac{1}{6}Y^3-\frac{67}{24}Y^2+\left(\frac{73}{9}-\frac{7}{12}\pi^2\right)Y-3\zeta_3\bigg]\frac{1}{xy}\Bigg\},
\end{aligned}
\tag{A.62}
$$

$$
\begin{aligned}
C_6^{[1]}=&\bigg[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-6\mathrm{Li}_4(-x)-8\mathrm{Li}_4(-y)+(4X+2Y)\mathrm{Li}_3(-x)+\frac{5}{24}X^4-\frac{3}{2}YX^3+\\
&+(1+X+8Y)\mathrm{Li}_3(-y)+\left(Y+XY-\frac{4}{3}\pi^2+3Y^2-2X^2\right)\mathrm{Li}_2(-x)-\frac{9}{4}X^3-\\
&-\left(\frac{7}{12}\pi^2-\frac{3}{2}x-\frac{9}{2}Y-2Y^2-\frac{7}{8}\right)X^2+\frac{13}{6}Y^3X-\left(3x-\frac{23}{4}+\frac{7}{6}\pi^2\right)YX+\\
&+\left(12-\zeta_3-\frac{7}{4}\pi^2\right)X-\left(\frac{11}{8}\pi^2-\frac{3}{2}x-\frac{3}{2}\right)Y^2-\left(\frac{93}{16}+\frac{17}{12}\pi^2-3\zeta_3\right)Y+\\
&+\frac{3}{2}\pi^2x-\frac{99}{16}\pi^2-\frac{19}{4}\zeta_3+\frac{44}{45}\pi^4+\frac{511}{64}\bigg]\frac{x}{y}+\\
&+\bigg[4\mathrm{Li}_3(-y)+4Y\mathrm{Li}_2(-x)+\left(\frac{3}{4}Y+\frac{\pi^2}{2}\right)X+\frac{3}{2}Y^2-\left(6-\frac{7}{4}\pi^2\right)Y-\frac{11}{12}\pi^2-\\
&\quad-4\zeta_3\bigg]\frac{1}{y}+\frac{11}{8}Y^2X-\left(\frac{21}{8}Y^2X-\frac{1}{3}Y^3\right)\frac{1}{x}+\\
&+\bigg[6\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)-Y\mathrm{Li}_3(-x)\right)+(3X+3-4Y)\mathrm{Li}_3(-y)+\frac{1}{6}Y^4+\\
&\quad+\frac{3}{2}Y^2X^2+(3Y+3XY-Y^2-2\pi^2)\mathrm{Li}_2(-x)-\left(\pi^2Y+3\zeta_3+\frac{3}{2}Y^3\right)X-\\
&\quad-\left(\frac{\pi^2}{8}-\frac{3}{2}\right)Y^2+\left(4\zeta_3-\frac{19}{12}\pi^2\right)Y-3\zeta_3\bigg]\frac{1}{xy}+\\
&+i\pi\Bigg\{\bigg[6\mathrm{Li}_3(-x)+9\mathrm{Li}_3(-y)+(1-3X+7Y)\mathrm{Li}_2(-x)-\left(\frac{3}{2}Y+\frac{9}{4}\right)X^2+
\end{aligned}
$$

$$
+\left(\frac{15}{2}+6Y^2-2\pi^2+\frac{21}{4}Y\right)X+\left(\frac{35}{4}-\frac{35}{12}\pi^2\right)Y+\frac{99}{16}-\frac{29}{12}\pi^2+2\zeta_3\bigg]\frac{x}{y}+
$$

$$
+\left[4\mathrm{Li}_2(-x)+\frac{3}{4}(1-2Y)X+\frac{\pi^2}{12}-6+\frac{15}{4}Y\right]\frac{1}{y}-\left(\frac{1}{8}Y^2+\frac{1}{6}Y^3\right)\frac{1}{x}+
$$

$$
+\bigg[-6\mathrm{Li}_3(-x)-\mathrm{Li}_3(-y)+(Y+3+3X)\mathrm{Li}_2(-x)+\left(Y^2+\frac{9}{4}Y\right)X+
$$

$$
+\left(3+\frac{\pi^2}{12}\right)Y+\zeta_3\bigg]\frac{1}{xy}+\frac{1}{6}Y^3+\frac{15}{8}Y^2\bigg\}, \tag{A.63}
$$

$$
D_6^{[1]} = \bigg[-\frac{1}{3}(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y)-(X-Y)\mathrm{Li}_2(-x))+\frac{1}{9}X^3+\frac{5}{18}Y^3-\left(\frac{29}{36}-\frac{1}{6}Y\right)X^2+
$$

$$
+\left(\frac{13}{72}\pi^2+\frac{31}{27}+\frac{11}{18}Y-\frac{2}{3}Y^2\right)X-\frac{37}{36}Y^2+\left(\frac{\pi^2}{3}+\frac{145}{54}\right)Y+\frac{55}{72}\pi^2-\frac{455}{108}-
$$

$$
-\frac{37}{36}\zeta_3\bigg]\frac{x}{y}+i\pi\left\{\frac{1}{3}X^2-\left(\frac{2}{3}Y+1\right)X+\frac{7}{24}\pi^2+\frac{23}{6}-\frac{13}{9}Y+\frac{1}{3}Y^2\right\}\frac{x}{y}, \tag{A.64}
$$

$$
E_6^{[1]} = \bigg[\frac{2}{3}(\mathrm{Li}_3(-x)+2\mathrm{Li}_3(-y)-(X-2Y)\mathrm{Li}_2(-x))-\frac{2}{9}X^3-\left(\frac{1}{3}Y-\frac{29}{18}\right)X^2+
$$

$$
+\left(\frac{5}{3}Y^2-\frac{13}{36}\pi^2-\frac{62}{27}-\frac{11}{9}Y\right)X-\left(\frac{101}{72}\pi^2-\frac{107}{27}\right)Y-\frac{\pi^2}{24}-\frac{59}{36}\zeta_3-\frac{685}{162}\bigg]\frac{x}{y}+
$$

$$
+\left(\frac{16}{9}Y+\frac{4}{9}\pi^2\right)\frac{1}{y}+\frac{2}{9}Y^3+\frac{1}{9}Y^2+(8Y^2-2Y^3)\frac{1}{9x}-\frac{4}{9xy}\pi^2Y+
$$

$$
+i\pi\left\{\left[\frac{2}{3}\mathrm{Li}_2(-x)-\frac{2}{3}X^2+\left(\frac{4}{3}Y+2\right)X+\frac{1}{3}Y^2-\frac{13}{9}Y+\frac{5}{3}-\frac{47}{72}\pi^2\right]\frac{x}{y}+\right.
$$

$$
\left.+\left(\frac{16}{9}-2Y\right)\frac{1}{y}-\left(\frac{16}{9}Y-\frac{2}{3}Y^2\right)\frac{1}{xy}\right\}, \tag{A.65}
$$

$$
F_6^{[1]} = \left[-\frac{1}{9}\pi^2+\frac{1}{9}Y^2-\frac{10}{27}Y+\frac{25}{81}\right]\frac{x}{y}+i\pi\left[\frac{2}{9}Y-\frac{10}{27}\right]\frac{x}{y}, \tag{A.66}
$$

For $h=6$ in eq. (2.16) and color factor $\mathrm{Tr}^{[2]}$ in eq. (2.21):

$$
G_6^{[2]} = \bigg[2\mathrm{Li}_4\left(-\frac{x}{y}\right)-4\mathrm{Li}_4(-x)-\frac{5}{2}\mathrm{Li}_4(-y)+\left(\frac{9}{2}+4X-Y\right)\mathrm{Li}_3(-x)-\frac{1}{12}YX^3+
$$

$$
+\left(Y+\frac{29}{6}+X\right)\mathrm{Li}_3(-y)+\left(\frac{3}{4}Y^2-\frac{\pi^2}{3}+\frac{29}{6}Y-\frac{9}{2}X-X^2\right)\mathrm{Li}_2(-x)+
$$

$$
+\left(\frac{3}{4}-\frac{9}{4}Y+\frac{1}{2}x+\frac{5}{8}Y^2-\frac{3}{8}\pi^2\right)X^2-\frac{1}{6}Y^3X+\frac{25}{6}Y^2X+\frac{1}{24}Y^4+\frac{1}{3}Y^3-
$$

$$
-\left(x+\frac{11}{8}+\frac{5}{6}\pi^2\right)YX-\left(\frac{5}{12}\pi^2+\zeta_3\right)X-\left(\frac{29}{36}-\frac{2}{3}\pi^2-\frac{1}{2}x\right)Y^2-
$$

$$
-\left(\frac{67}{18}\pi^2-2\zeta_3-\frac{3049}{432}\right)Y+\frac{115}{72}\pi^2-\frac{23213}{5184}+\frac{\pi^2}{2}x-\frac{341}{72}\zeta_3+\frac{59}{160}\pi^4\bigg]\frac{x}{y}+
$$

$$
+\bigg[-3\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)+\mathrm{Li}_4(-y)-\mathrm{Li}_3(-x)+\left(X-Y-\frac{1}{2}\right)\mathrm{Li}_3(-y)\right)-
$$

$$
\begin{aligned}
&-\left(3X-\frac{3}{2}Y-\frac{\pi^2}{2}\right)\mathrm{Li}_2(-x)-\frac{3}{4}\left(Y^2+2Y\right)X^2+\frac{1}{2}Y^3X+\frac{7}{4}Y^2X-\\
&-\frac{1}{3}Y^3+\left(\pi^2-\frac{3}{2}\right)YX-\left(\frac{\pi^2}{6}-3\zeta_3\right)X+\frac{1}{4}(\pi^2+1)Y^2-\\
&-\left(3\zeta_3+\frac{13}{12}\pi^2-\frac{32}{9}\right)Y+\frac{59}{36}\pi^2+\frac{\pi^4}{30}-2\zeta_3\Bigg]\frac{1}{y}+\\
&+\left[-3\mathrm{Li}_4\left(-\frac{x}{y}\right)-3\mathrm{Li}_4(-x)+\frac{1}{2}\mathrm{Li}_4(-y)+\left(\frac{3}{2}-3X+Y\right)\mathrm{Li}_3(-y)-\frac{3}{4}Y^2X^2-\right.\\
&-\frac{1}{12}Y^4-\left(\frac{1}{4}Y^2-\frac{\pi^2}{6}-\frac{3}{2}Y\right)\mathrm{Li}_2(-x)+\left(\frac{3}{2}Y^2+\frac{2}{3}\pi^2Y+3\zeta_3+\frac{1}{4}Y^3\right)X+\\
&\left.+\frac{1}{9}Y^3-\left(\frac{37}{36}-\frac{3}{8}\pi^2\right)Y^2-\left(\zeta_3+\frac{31}{18}\pi^2\right)Y-\frac{\pi^4}{180}-\frac{3}{2}\zeta_3\right]\frac{1}{xy}+\\
&+i\pi\left\{\left[3\mathrm{Li}_3(-x)+2\mathrm{Li}_3(-y)+\left(\frac{1}{3}+\frac{3}{2}Y-2X\right)\mathrm{Li}_2(-x)-\frac{1}{12}X^3+\frac{1}{2}X^2Y-\right.\right.\\
&\qquad-\left(Y+\frac{13}{12}\pi^2-\frac{1}{8}\right)X-\frac{5}{12}Y^3+\frac{11}{4}Y^2-\left(\frac{215}{72}+\frac{\pi^2}{3}\right)Y+\\
&\qquad\left.+\frac{3049}{432}+\zeta_3+\frac{\pi^2}{36}\right]\frac{x}{y}-\frac{1}{2}Y^2X+\frac{1}{2x}XY^2+\\
&\qquad+\left[-\frac{3}{2}\mathrm{Li}_2(-x)-\left(\frac{3}{2}+Y\right)X+\frac{\pi^2}{12}+\frac{32}{9}-Y\right]\frac{1}{y}+\\
&\qquad+\left[-2\mathrm{Li}_3(-y)-\frac{1}{2}(Y-3)\mathrm{Li}_2(-x)+\frac{3}{2}XY-\frac{1}{6}Y^3+\frac{13}{12}Y^2-\right.\\
&\qquad\left.\left.-\left(\frac{37}{18}-\frac{\pi^2}{4}\right)Y+2\zeta_3\right]\frac{1}{xy}\right\},
\end{aligned}
\tag{A.67}
$$

$$
\begin{aligned}
H_6^{[2]}=&\left[-\mathrm{Li}_4\left(-\frac{x}{y}\right)+6\mathrm{Li}_4(-x)+4\mathrm{Li}_4(-y)-\left(5X-\frac{1}{3}\right)\mathrm{Li}_3(-x)+\frac{1}{24}X^4-\frac{1}{12}Y^4-\right.\\
&-\left(X-\frac{13}{6}+4Y\right)\mathrm{Li}_3(-y)+\left(\frac{13}{6}Y+\frac{3}{2}\pi^2-\frac{1}{3}X-\frac{5}{2}Y^2+\frac{3}{2}X^2\right)\mathrm{Li}_2(-x)+\\
&+\left(\frac{1}{12}Y-\frac{11}{18}\right)X^3+\left(\frac{1}{2}x+\frac{17}{24}\pi^2-\frac{3}{8}Y^2+\frac{7}{12}Y+\frac{277}{72}\right)X^2-\frac{17}{12}Y^3X+\\
&+\frac{47}{24}Y^2X+\left(\frac{5}{6}\pi^2-\frac{347}{72}-x\right)YX-\left(\frac{79}{27}+\frac{143}{144}\pi^2-\frac{1}{2}\zeta_3\right)X+\\
&+\left(\frac{1}{2}x+\frac{5}{6}\pi^2-\frac{29}{9}\right)Y^2-\left(\frac{3}{2}\zeta_3+\frac{215}{144}\pi^2-\frac{781}{54}\right)Y-\frac{1031}{1440}\pi^4+\frac{467}{144}\pi^2+\\
&\left.+\frac{\pi^2}{2}x+\frac{311}{72}\zeta_3-\frac{30659}{1296}\right]\frac{x}{y}+\\
&+\left[-\mathrm{Li}_3(-y)-Y\mathrm{Li}_2(-x)+\left(\frac{5}{4}Y-\frac{\pi^2}{2}+\frac{1}{4}Y^2\right)X-\frac{21}{4}Y^2-\left(\frac{9}{4}\pi^2-\frac{59}{9}\right)Y+\right.\\
&\left.+\zeta_3+\frac{8}{9}\pi^2\right]\frac{1}{y}+\frac{13}{36}Y^3-\frac{31}{36x}Y^3+
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-2\left(\mathrm{Li}_4\left(-\frac{x}{y}\right)+\mathrm{Li}_4(-x)-Y\mathrm{Li}_3(-x)\right)-\left(X+\frac{5}{2}-Y\right)\mathrm{Li}_3(-y)-\frac{1}{24}Y^4-\right.\\
&\quad-\left(\frac{5}{2}Y+XY-\pi^2\right)\mathrm{Li}_2(-x)-\frac{1}{2}Y^2X^2+\left(\frac{2}{3}\pi^2Y+\zeta_3+\frac{1}{6}Y^3-\frac{19}{8}Y^2\right)X+\\
&\quad\left.+\left(\frac{11}{24}\pi^2-\frac{127}{36}\right)Y^2-\left(\zeta_3-\frac{19}{36}\pi^2\right)Y+\frac{5}{2}\zeta_3\right]\frac{1}{xy}+\\
&+i\pi\left\{\left[-5(\mathrm{Li}_3(-x)+\mathrm{Li}_3(-y))-\left(5Y-\frac{11}{6}-3X\right)\mathrm{Li}_2(-x)+\frac{1}{12}X^3-\frac{1}{4}Y^3-\right.\right.\\
&\qquad-\frac{13}{12}X^2+\left(\frac{35}{12}Y+\frac{23}{8}-\frac{5}{2}Y^2+\frac{7}{4}\pi^2\right)X-\frac{5}{24}Y^2-\left(\frac{811}{72}-\frac{3}{2}\pi^2\right)Y+\\
&\qquad\left.+\frac{623}{54}-\zeta_3-\frac{85}{72}\pi^2\right]\frac{x}{y}+\\
&\qquad+\left[-\mathrm{Li}_2(-x)+\left(\frac{5}{4}+\frac{3}{2}Y\right)X-\frac{\pi^2}{4}+\frac{9}{4}Y^2-\frac{37}{4}Y+\frac{59}{9}\right]\frac{1}{y}+\\
&\qquad+\left[2\mathrm{Li}_3(-x)-\left(Y+X+\frac{5}{2}\right)\mathrm{Li}_2(-x)-\left(Y^2+\frac{9}{4}Y\right)X-\frac{1}{6}Y^3+\frac{35}{24}Y^2-\right.\\
&\qquad\left.\left.-\left(\frac{127}{18}-\frac{\pi^2}{4}\right)Y\right]\frac{1}{xy}\right\},
\end{aligned}
\tag{A.68}
$$

$$
\begin{aligned}
I_6^{[2]}=&\left[\mathrm{Li}_4\left(-\frac{x}{y}\right)+2\mathrm{Li}_4(-x)+\frac{5}{2}\mathrm{Li}_4(-y)-(Y+X)\mathrm{Li}_3(-x)-\left(3Y+\frac{1}{2}\right)\mathrm{Li}_3(-y)+\right.\\
&+\left(-\frac{5}{4}Y^2-\frac{1}{2}Y+\frac{1}{2}X^2+\frac{\pi^2}{2}\right)\mathrm{Li}_2(-x)-\frac{1}{12}X^4-\frac{11}{12}Y^3X+\left(\frac{3}{4}+\frac{1}{2}Y\right)X^3-\\
&-\left(\frac{3}{2}Y+\frac{1}{2}Y^2-\frac{\pi^2}{6}+\frac{1}{2}x-\frac{3}{8}\right)X^2+\left(x-\frac{13}{4}+\frac{\pi^2}{3}\right)YX-\left(6-\frac{7}{12}\pi^2\right)X+\\
&\left.+\frac{7}{12}\pi^2Y^2+\left(\frac{2}{3}\pi^2-3\zeta_3+\frac{45}{16}\right)Y-\frac{511}{64}-\frac{\pi^2}{2}x-\frac{109}{360}\pi^4+\frac{15}{4}\zeta_3+\frac{39}{16}\pi^2\right]\frac{x}{y}+\\
&+\left[-\frac{3}{2}\mathrm{Li}_3(-y)-\frac{3}{2}Y\mathrm{Li}_2(-x)-\left(\frac{1}{4}Y+\frac{\pi^2}{6}\right)X-Y^2+\left(3-\frac{2}{3}\pi^2\right)Y+\zeta_3+\right.\\
&\quad\left.+\frac{5}{12}\pi^2\right]\frac{1}{y}+\frac{1}{2}xY^2-\frac{3}{8}Y^2X+\left(\frac{7}{8}Y^2X-\frac{1}{6}Y^3\right)\frac{1}{x}+\\
&+\left[-2\mathrm{Li}_4\left(-\frac{x}{y}\right)-2\mathrm{Li}_4(-x)+\frac{1}{2}\mathrm{Li}_4(-y)+2Y\mathrm{Li}_3(-x)+(Y-1-X)\mathrm{Li}_3(-y)+\right.\\
&\quad+\left(-Y+\frac{2}{3}\pi^2-YX+\frac{1}{4}Y^2\right)\mathrm{Li}_2(-x)-\frac{1}{2}Y^2X^2+\left(\frac{\pi^2}{3}Y+\frac{5}{12}Y^3+\zeta_3\right)X-\\
&\quad\left.-\frac{1}{24}Y^4-Y^2+\left(\frac{5}{12}\pi^2-\zeta_3\right)Y-\frac{\pi^4}{180}+\zeta_3\right]\frac{1}{xy}+\\
&+i\pi\left\{\left[-2\mathrm{Li}_3(-x)-3\mathrm{Li}_3(-y)-\left(\frac{5}{2}Y+\frac{1}{2}-X\right)\mathrm{Li}_2(-x)+\left(\frac{3}{4}+\frac{1}{2}Y\right)X^2+\right.\right.\\
&\qquad\left.+\left(\frac{2}{3}\pi^2-\frac{5}{2}-2Y^2-\frac{7}{4}Y\right)X+\left(\pi^2-\frac{17}{4}\right)Y-3\zeta_3+\pi^2-\frac{51}{16}\right]\frac{x}{y}+
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{3}{2}\mathrm{Li}_2(-x)+\left(\frac{1}{2}Y-\frac{1}{4}\right)X+3-\frac{9}{4}Y\right]\frac{1}{y}-\frac{5}{8}Y^2-\frac{1}{8x}Y^2+\\
&+\left[2\mathrm{Li}_3(-x)-\left(X+1+\frac{1}{2}Y\right)\mathrm{Li}_2(-x)-\left(\frac{1}{2}Y+\frac{3}{4}\right)YX-2Y\right]\frac{1}{xy}\Bigg\},
\end{aligned}
\tag{A.69}
$$

$$
\begin{aligned}
J_6^{[2]} =& \left[-\frac{1}{3}\mathrm{Li}_3(-y)-\frac{1}{3}Y\mathrm{Li}_2(-x)-\frac{1}{6}Y^2X-\frac{1}{6}Y^3+\frac{13}{12}Y^2-\left(\frac{193}{54}-\frac{7}{18}\pi^2\right)Y+\frac{455}{108}+\right.\\
&\left.+\frac{49}{36}\zeta_3-\frac{19}{24}\pi^2\right]\frac{x}{y}-\left(\frac{8}{9}Y+\frac{2}{9}\pi^2-\frac{1}{2}Y^2\right)\frac{1}{y}-(Y^3-4Y^2-2\pi^2Y)\frac{1}{9xy}+\\
&+i\pi\Bigg\{\left[-\frac{1}{2}Y^2+\frac{13}{6}Y-\frac{1}{3}\mathrm{Li}_2(-x)-\frac{193}{54}+\frac{\pi^2}{18}\right]\frac{x}{y}-\left(\frac{8}{9}-Y\right)\frac{1}{y}+\\
&+\left(\frac{8}{9}Y-\frac{1}{3}Y^2\right)\frac{1}{xy}\Bigg\},
\end{aligned}
\tag{A.70}
$$

$$
\begin{aligned}
K_6^{[2]} =& \left[-\frac{1}{3}\mathrm{Li}_3(-x)-\frac{2}{3}\mathrm{Li}_3(-y)+\frac{1}{3}(X-2Y)\mathrm{Li}_2(-x)+\frac{1}{9}X^3-\left(\frac{29}{36}-\frac{1}{6}Y\right)X^2-\right.\\
&\left.-\left(\frac{5}{6}Y^2-\frac{13}{72}\pi^2-\frac{31}{27}-\frac{11}{18}Y\right)X+\left(\frac{49}{72}\pi^2-\frac{83}{27}\right)Y+\frac{5}{72}\pi^2+\frac{47}{36}\zeta_3+\frac{685}{162}\right]\frac{x}{y}-\\
&-(8Y+2\pi^2)\frac{1}{9y}-\frac{1}{9}Y^3-\frac{1}{18}Y^2-(4Y^2-Y^3)\frac{1}{9x}+\frac{2\pi^2}{9xy}Y+\\
&+i\pi\Bigg\{\left[-\frac{1}{3}\mathrm{Li}_2(-x)+\frac{1}{3}X^2-\left(\frac{2}{3}Y+1\right)X-\frac{1}{6}Y^2+\frac{13}{18}Y-\frac{52}{27}+\frac{11}{36}\pi^2\right]\frac{x}{y}-\\
&-\left(\frac{8}{9}-Y\right)\frac{1}{y}+\left(\frac{8}{9}Y-\frac{1}{3}Y^2\right)\frac{1}{xy}\Bigg\},
\end{aligned}
\tag{A.71}
$$

$$
L_6^{[2]} = \left[\frac{\pi^2}{9}-\frac{1}{9}Y^2+\frac{10}{27}Y-\frac{25}{81}\right]\frac{x}{y}+i\pi\left[-\frac{2}{9}Y+\frac{10}{27}\right]\frac{x}{y}, \tag{A.72}
$$

## B. Auxiliary functions for two-loop scheme shifts

In this appendix we present auxiliary functions appearing in eq. (5.10) for the shift in the two-loop amplitudes under scheme changes. These functions correspond to the $\delta_R$-dependent parts of the $\mathcal{O}(\epsilon)$ terms in the one-loop amplitude remainders. They are given by,

$$
M_1^{(1),[1]\epsilon,\delta_R} = -\frac{19}{18}yN-\left(Y-\frac{y}{2}\right)\frac{1}{N}-i\pi\left[\frac{y}{3}N+(x+3)\frac{1}{2N}\right], \tag{B.1}
$$

$$
M_1^{(1),[2]\epsilon,\delta_R} = \frac{1}{2}Y+\frac{19}{18}y+(Y-y)\frac{1}{2N^2}+i\pi\left[-\frac{x}{3}+\frac{1}{6}+\left(1+\frac{1}{2}x\right)\frac{1}{N^2}\right], \tag{B.2}
$$

$$
\begin{aligned}
M_2^{(1),[1]\epsilon,\delta_R} =& -\frac{19}{18}xN+\left(-\frac{1}{2x}Y^2+Y+\frac{x}{2}\right)\frac{1}{N}+\\
&+i\pi\left[-\frac{x}{3}N+\left(-\frac{1}{x}Y+\frac{x}{2}+1\right)\frac{1}{N}\right],
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
M_2^{(1),[2]\epsilon,\delta_R} &= \frac{1}{4x}Y^2 - \frac{1}{2}Y + \frac{19}{18}x + \left(\frac{1}{4x}Y^2 - \frac{1}{2}Y - \frac{x}{2}\right)\frac{1}{N^2} + \\
&\quad + i\pi\left[\frac{1}{2x}Y + \frac{x}{3} - \frac{1}{2} + \left(\frac{1}{x}Y + y\right)\frac{1}{2N^2}\right], \qquad \text{(B.4)} \\
M_3^{(1),[1]\epsilon,\delta_R} &= \left[-\frac{\pi^2}{2}y + X - \frac{y}{2}(X-Y)^2 + \frac{1}{2y} - \left(\frac{1}{2y}+1\right)Y\right]\frac{1}{N} + \\
&\quad + \left(\frac{1}{3}Y - \frac{19}{18}\right)\frac{N}{y}, \qquad \text{(B.5)} \\
M_3^{(1),[2]\epsilon,\delta_R} &= \left[\frac{\pi^2}{4}y - \frac{1}{2}X + 1 + \frac{y}{4}(X-Y)^2\right]\left(1 + \frac{1}{N^2}\right) + \left(x + \frac{37}{18}\right)\frac{1}{y} + \\
&\quad + \left(x + \frac{1}{2} - \frac{x}{2}Y\right)\frac{1}{yN^2} + \left(\frac{1}{2} - \frac{1}{3y}\right)Y, \qquad \text{(B.6)} \\
M_4^{(1),[1]\epsilon,\delta_R} &= \left[-X + \left(\frac{3}{2} + \frac{1}{2y}\right)Y + \frac{x}{2y}\right]\frac{1}{N} + \left(\frac{1}{3}Y - \frac{19}{18}\right)\frac{x}{y}N, \qquad \text{(B.7)} \\
M_4^{(1),[2]\epsilon,\delta_R} &= \left[\frac{1}{2}X + \frac{1}{y}\left(x + \frac{1}{2}\right)Y - \frac{x}{2y}\right]\frac{1}{N^2} + \frac{1}{2}X + \frac{1}{6y}(x+3)Y + \frac{19}{18}\frac{x}{y}, \qquad \text{(B.8)} \\
M_5^{(1),[1]\epsilon,\delta_R} &= \left[\left(1 - \frac{1}{2y}\right)Y + \frac{1}{2y}\right]\frac{1}{N} + \left(\frac{1}{3}Y - \frac{19}{18}\right)\frac{N}{y} + i\pi\frac{1}{N}, \qquad \text{(B.9)} \\
M_5^{(1),[2]\epsilon,\delta_R} &= \left[\left(\frac{x}{2} + 1\right)Y - \frac{1}{2}\right]\frac{1}{yN^2} - \left(\frac{1}{3y} + \frac{1}{2}\right)Y + \frac{19}{18y} - i\frac{\pi}{2}\left(1 + \frac{1}{N^2}\right), \qquad \text{(B.10)} \\
M_6^{(1),[1]\epsilon,\delta_R} &= \left[-\frac{y}{2x}Y^2 + \frac{1}{y}\left(\frac{x}{2} + 1\right)Y + \frac{x}{2y}\right]\frac{1}{N} + \left(\frac{1}{3}Y - \frac{19}{18}\right)\frac{x}{y}N - \\
&\quad - i\pi\left(\frac{y}{x}Y + 1\right)\frac{1}{N}, \qquad \text{(B.11)} \\
M_6^{(1),[2]\epsilon,\delta_R} &= \left[\frac{y}{4x}Y^2 - \frac{1}{2y}Y - \frac{x}{2y}\right]\frac{1}{N^2} + \frac{y}{4x}Y^2 + \left(\frac{1}{2} - \frac{x}{3y}\right)Y + \frac{19}{18}\frac{x}{y} + \\
&\quad + i\frac{\pi}{2}\left(\frac{y}{x}Y + 1\right)\left(1 + \frac{1}{N^2}\right). \qquad \text{(B.12)}
\end{aligned}
$$

## References


[1] F. Aversa, P. Chiappetta, M. Greco and J.P. Guillet, *Higher order corrections to QCD jets*, *Phys. Lett.* **B 210** (1988) 225; *Jet production in hadronic collisions to $\mathcal{O}(\alpha_s^3)$*, *Z. Physik* **C 46** (1990) 253.

[2] S.D. Ellis, Z. Kunszt and D.E. Soper, *The one jet inclusive cross-section at $\mathcal{O}(\alpha_s^3)$ quarks and gluons*, *Phys. Rev. Lett.* **64** (1990) 2121; *Two jet production in hadron collisions at $\mathcal{O}(\alpha_s^3)$ in QCD*, *Phys. Rev. Lett.* **69** (1992) 1496.

[3] W.T. Giele, E.W.N. Glover and D.A. Kosower, *Higher order corrections to jet cross-sections in hadron colliders*, *Nucl. Phys.* **B 403** (1993) 633 [hep-ph/9302225].

[4] N. Kidonakis, *Resummation for heavy quark and jet cross sections*, *Int. J. Mod. Phys.* **A 15** (2000) 1245 [hep-ph/9902484];
N. Kidonakis and J.F. Owens, *Effects of higher-order threshold corrections in high-$E_T$ jet production*, *Phys. Rev.* **D 63** (2001) 054019 [hep-ph/0007268].

[5] S. Alekhin, *Extraction of parton distributions and $\alpha_s$ from DIS data within the bayesian treatment of systematic errors*, *Eur. Phys. J.* **C 10** (1999) 395;
W.T. Giele, S.A. Keller and D.A. Kosower, *Parton distributions with errors*, in La Thuile 1999, *Results and perspectives in particle physics*;
D. Stump et al., *Uncertainties of predictions from parton distribution functions, I. The Lagrange multiplier method*, *Phys. Rev.* **D 65** (2002) 014012 [hep-ph/0101051];
J. Pumplin et al., *Uncertainties of predictions from parton distribution functions II. The hessian method*, *Phys. Rev.* **D 65** (2002) 014013 [hep-ph/0101032];
W.T. Giele, S.A. Keller and D.A. Kosower, *Parton distribution function uncertainties*, hep-ph/0104052;
A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, *Uncertainties of predictions from parton distributions, I. Experimental errors*, *Eur. Phys. J.* **C 28** (2003) 455 [hep-ph/0211080].

[6] S. Moch, J.A.M. Vermaseren and A. Vogt, *The three-loop splitting functions in QCD: the non-singlet case*, *Nucl. Phys.* **B 688** (2004) 101 [hep-ph/0403192];
A. Vogt, S. Moch and J.A.M. Vermaseren, *The three-loop splitting functions in QCD: the singlet case*, *Nucl. Phys.* **B 691** (2004) 129 [hep-ph/0404111].

[7] S. Alekhin, *Parton distributions from deep-inelastic scattering data*, *Phys. Rev.* **D 68** (2003) 014002 [hep-ph/0211096];
A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, *NNLO global parton analysis*, *Phys. Lett.* **B 531** (2002) 216 [hep-ph/0201127].

[8] W.L. van Neerven and A. Vogt, *NNLO evolution of deep-inelastic structure functions: the non-singlet case*, *Nucl. Phys.* **B 568** (2000) 263 [hep-ph/9907472]; *NNLO evolution of deep-inelastic structure functions: the singlet case*, *Nucl. Phys.* **B 588** (2000) 345 [hep-ph/0006154];
*Improved approximations for the three-loop splitting functions in QCD*, *Phys. Lett.* **B 490** (2000) 111 [hep-ph/0007362];
S. Moch, J.A.M. Vermaseren and A. Vogt, *Non-singlet structure functions at three loops: fermionic contributions*, *Nucl. Phys.* **B 646** (2002) 181 [hep-ph/0209100].

[9] E.W.N. Glover, *Progress in NNLO calculations for scattering processes*, *Nucl. Phys.* **116** *(Proc. Suppl.)* (2003) 3 [hep-ph/0211412].

[10] Z. Bern, J.S. Rozowsky and B. Yan, *Two-loop four-gluon amplitudes in $N = 4$ super-Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424];
Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys.* **B 530** (1998) 401 [hep-th/9802162].

[11] Z. Bern, L.J. Dixon and D.A. Kosower, *A two-loop four-gluon helicity amplitude in QCD*, *J. High Energy Phys.* **01** (2000) 027 [hep-ph/0001001].

[12] Z. Bern, L.J. Dixon and A. Ghinculov, *Two-loop correction to Bhabha scattering*, *Phys. Rev.* **D 63** (2001) 053007 [hep-ph/0010075].

[13] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two-loop QCD corrections to $q\bar{q} \to q'\bar{q}'$*, *Nucl. Phys.* **B 601** (2001) 318 [hep-ph/0010212]; *Two-loop QCD corrections to $q\bar{q} \to q\bar{q}$*, *Nucl. Phys.* **B 601** (2001) 341 [hep-ph/0011094].

[14] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two-loop QCD corrections to massless quark gluon scattering*, *Nucl. Phys.* **B 605** (2001) 486 [hep-ph/0101304].

[15] E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two-loop QCD corrections to gluon gluon scattering*, *Nucl. Phys.* **B 605** (2001) 467 [hep-ph/0102201].

[16] Z. Bern, A. De Freitas and L.J. Dixon, *Two-loop amplitudes for gluon fusion into two photons*, *J. High Energy Phys.* **09** (2001) 037 [hep-ph/0109078].

[17] Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov and H.L. Wong, *QCD and qed corrections to light-by-light scattering*, *J. High Energy Phys.* **11** (2001) 031 [hep-ph/0109079].

[18] Z. Bern, A. De Freitas and L.J. Dixon, *Two-loop helicity amplitudes for gluon gluon scattering in QCD and supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **03** (2002) 018 [hep-ph/0201161].

[19] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, *The two-loop QCD matrix element for $e^+e^- \to$ 3jets*, *Nucl. Phys.* **B 627** (2002) 107 [hep-ph/0112081].

[20] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, *Two-loop QCD helicity amplitudes for $e^+e^- \to$ 3jets*, *Nucl. Phys.* **B 642** (2002) 227 [hep-ph/0206067];
S. Moch, P. Uwer and S. Weinzierl, *Two-loop amplitudes for $e^+e^- \to q\bar{q}g$: the $N_f$-contribution*, *Acta Phys. Polon.* **B33** (2002) 2921 [hep-ph/0207167].

[21] Z. Bern, A. De Freitas and L.J. Dixon, *Two-loop helicity amplitudes for quark gluon scattering in QCD and gluino gluon scattering in supersymmetric yang- mills theory*, *J. High Energy Phys.* **06** (2003) 028 [hep-ph/0304168].

[22] E.W.N. Glover and M.E. Tejeda-Yeomans, *Two-loop QCD helicity amplitudes for massless quark massless gauge boson scattering*, *J. High Energy Phys.* **06** (2003) 033 [hep-ph/0304169].

[23] E.W.N. Glover, *Two-loop QCD helicity amplitudes for massless quark quark scattering*, *J. High Energy Phys.* **04** (2004) 021 [hep-ph/0401119].

[24] F.V. Tkachov, *A theorem on analytical calculability of four loop renormalization group functions*, *Phys. Lett.* **B 100** (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, *Integration by parts: the algorithm to calculate beta functions in 4 loops*, *Nucl. Phys.* **B 192** (1981) 159.

[25] V.A. Smirnov, *Analytical result for dimensionally regularized massless on-shell double box*, *Phys. Lett.* **B 460** (1999) 397 [hep-ph/9905323].

[26] J.B. Tausk, *Non-planar massless two-loop feynman diagrams with four on- shell legs*, *Phys. Lett.* **B 469** (1999) 225 [hep-ph/9909506].

[27] T. Gehrmann and E. Remiddi, *Differential equations for two-loop four-point functions*, *Nucl. Phys.* **B 580** (2000) 485 [hep-ph/9912329].

[28] C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, *The tensor reduction and master integrals of the two-loop massless crossed box with light-like legs*, *Nucl. Phys.* **B 580** (2000) 577 [hep-ph/0003261].

[29] V.A. Smirnov and O.L. Veretin, *Analytical results for dimensionally regularized massless on-shell double boxes with arbitrary indices and numerators*, *Nucl. Phys.* **B 566** (2000) 469 [hep-ph/9907385].

[30] C. Anastasiou, E.W.N. Glover and C. Oleari, *Application of the negative-dimension approach to massless scalar box integrals*, *Nucl. Phys.* **B 565** (2000) 445 [hep-ph/9907523]; *The two-loop scalar and tensor pentabox graph with light-like legs*, *Nucl. Phys.* **B 575** (2000) 416 [hep-ph/9912251].

[31] S. Catani, *The singular behaviour of QCD amplitudes at two-loop order*, *Phys. Lett.* **B 427** (1998) 161 [hep-ph/9802439].

[32] F.A. Berends and W. Giele, *The six gluon process as an example of Weyl-Van der Waerden spinor calculus*, *Nucl. Phys.* **B 294** (1987) 700;
M.L. Mangano, S.J. Parke and Z. Xu, *Duality and multi-gluon scattering*, *Nucl. Phys.* **B 298** (1988) 653.

[33] M.L. Mangano and S.J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301.

[34] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop corrections to five gluon amplitudes*, *Phys. Rev. Lett.* **70** (1993) 2677 [hep-ph/9302280];
Z. Kunszt, A. Signer and Z. Trócsányi, *One loop radiative corrections to the helicity amplitudes of QCD processes involving four quarks and one gluon*, *Phys. Lett.* **B 336** (1994) 529 [hep-ph/9405386];
Z. Bern, L.J. Dixon and D.A. Kosower, *One loop corrections to two quark three gluon amplitudes*, *Nucl. Phys.* **B 437** (1995) 259 [hep-ph/9409393].

[35] J. Soffer and J.M. Virey, *Testing various polarized parton distributions at RHIC*, *Nucl. Phys.* **B 509** (1998) 297 [hep-ph/9706229].

[36] D. de Florian, S. Frixione, A. Signer and W. Vogelsang, *Next-to-leading order jet cross sections in polarized hadronic collisions*, *Nucl. Phys.* **B 539** (1999) 455 [hep-ph/9808262].

[37] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, hep-th/0312171.

[38] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *J. High Energy Phys.* **09** (2004) 006 [hep-th/0403047].

[39] I. Bena, Z. Bern and D.A. Kosower, *Twistor-space recursive formulation of gauge theory amplitudes*, hep-th/0406133.

[40] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *J. High Energy Phys.* **05** (2004) 070 [hep-th/0404072];
G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *J. High Energy Phys.* **07** (2004) 048 [hep-th/0407027].

[41] F. Cachazo, P. Svrcek and E. Witten, *Twistor space structure of one-loop amplitudes in gauge theory*, hep-th/0406177;
D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, hep-th/0406175;
A. Brandhuber, B. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, hep-th/0407214.

[42] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, *Supergravity and the S matrix*, *Phys. Rev.* **D 15** (1977) 996; *Some properties of scattering amplitudes in supersymmetric theories*, *Nucl. Phys.* **B 124** (1977) 81;
S.J. Parke and T.R. Taylor, *Perturbative QCD utilizing extended supersymmetry*, *Phys. Lett.* **B 157** (1985) 81.


JHEP09(2004)039

[43] Z. Bern, A. De Freitas, L.J. Dixon and H.L. Wong, *Supersymmetric regularization, two-loop QCD amplitudes and coupling shifts*, *Phys. Rev.* **D 66** (2002) 085002 [hep-ph/0202271].

[44] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n-point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[45] Z. Bern, L.J. Dixon and D.A. Kosower, *Progress in one-loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109–148 [hep-ph/9602280].

[46] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *Multi-reggeon processes in the Yang-Mills theory*, *Sov. Phys. JETP* **44** (1976) 443.

[47] V. Del Duca and E.W.N. Glover, *The high energy limit of QCD at two loops*, *J. High Energy Phys.* **10** (2001) 035 [hep-ph/0109028];
A.V. Bogdan, V. Del Duca, V.S. Fadin and E.W.N. Glover, *The quark Regge trajectory at two loops*, *J. High Energy Phys.* **03** (2002) 032 [hep-ph/0201240].

[48] G. Sterman and M.E. Tejeda-Yeomans, *Multi-loop amplitudes and resummation*, *Phys. Lett.* **B 552** (2003) 48 [hep-ph/0210130].

[49] Z. Bern, L.J. Dixon and D.A. Kosower, *Two-loop $g \to gg$ splitting amplitudes in QCD*, *J. High Energy Phys.* **08** (2004) 012 [hep-ph/0404293].

[50] J.C. Collins, *Renormalization: an introduction to renormalization group, and the operator-product expansion*, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, 1984.

[51] R.K. Ellis and J.C. Sexton, *QCD radiative corrections to parton parton scattering*, *Nucl. Phys.* **B 269** (1986) 445.

[52] G. 't Hooft and M.J.G. Veltman, *Regularization and renormalization of gauge fields*, *Nucl. Phys.* **B 44** (1972) 189.

[53] Z. Bern and D.A. Kosower, *The computation of loop amplitudes in gauge theories*, *Nucl. Phys.* **B 379** (1992) 451.

[54] W. Siegel, *Supersymmetric dimensional regularization via dimensional reduction*, *Phys. Lett.* **B 84** (1979) 193;
D.M. Capper, D.R.T. Jones and P. van Nieuwenhuizen, *Regularization by dimensional reduction of supersymmetric and nonsupersymmetric gauge theories*, *Nucl. Phys.* **B 167** (1980) 479;
I. Jack, D.R.T. Jones and K.L. Roberts, *Equivalence of dimensional reduction and dimensional regularization*, *Z. Physik* **C 63** (1994) 151 [hep-ph/9401349].

[55] M.S. Chanowitz, M. Furman and I. Hinchliffe, *The axial current in dimensional regularization*, *Nucl. Phys.* **B 159** (1979) 225.

[56] H.F. Contopanagos and M.B. Einhorn, *Theory of the asymptotic S matrix for massless particles*, *Phys. Rev.* **D 45** (1992) 1291.

[57] F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T.T. Wu, *Single bremsstrahlung processes in gauge theories*, *Phys. Lett.* **B 103** (1981) 124;
P. De Causmaecker, R. Gastmans, W. Troost and T.T. Wu, *Helicity amplitudes for massless qed*, *Phys. Lett.* **B 105** (1981) 215;
Z. Xu, D.H. Zhang and L. Chang, *Helicity amplitudes for multiple bremsstrahlung in massless nonabelian gauge theories*, *Nucl. Phys.* **B 291** (1987) 392.

[58] Z. Kunszt, A. Signer and Z. Trócsányi, *Singular terms of helicity amplitudes at one loop in QCD and the soft limit of the cross-sections of multiparton processes*, *Nucl. Phys.* **B 420** (1994) 550 [hep-ph/9401294].

[59] C. Anastasiou, E.W.N. Glover and M.E. Tejeda-Yeomans, *Two-loop QED and QCD corrections to massless fermion boson scattering*, *Nucl. Phys.* **B 629** (2002) 255 [hep-ph/0201274].

[60] Z. Kunszt, A. Signer and Z. Trócsányi, *One loop helicity amplitudes for all $2 \to 2$ processes in QCD and $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 411** (1994) 397 [hep-ph/9305239].

[61] P. Nogueira, *Automatic Feynman graph generation*, *J. Comput. Phys.* **105** (1993) 279.

[62] L. Lewin, *Dilogarithms and associated functions*, Macdonald, 1958.

[63] K.S. Kölbig, J.A. Mignaco and E. Remiddi, *On Nielsen's generalized polylogarithms and their numerical calculation*, B.I.T. **10** (1970) 38.

[64] E.I. Gates and K.L. Kowalski, *Majorana Feynman rules*, *Phys. Rev.* **D 37** (1988) 938;
A. Denner, H. Eck, O. Hahn and J. Kublbeck, *Compact Feynman rules for Majorana fermions*, *Phys. Lett.* **B 291** (1992) 278.

[65] Z. Bern, L.J. Dixon and C. Schmidt, *Isolating a light Higgs boson from the di-photon background at the LHC*, *Phys. Rev.* **D 66** (2002) 074018 [hep-ph/0206194];
L.J. Dixon and M.S. Siu, *Resonance-continuum interference in the di-photon Higgs signal at the LHC*, *Phys. Rev. Lett.* **90** (2003) 252001 [hep-ph/0302233].

[66] G. Heinrich, *A numerical method for NNLO calculations*, *Nucl. Phys.* **116** *(Proc. Suppl.)* (2003) 368 [hep-ph/0211144];
D.A. Kosower, *Multiple singular emission in gauge theories*, *Phys. Rev.* **D 67** (2003) 116003 [hep-ph/0212097];
S. Weinzierl, *Subtraction terms at NNLO*, *J. High Energy Phys.* **03** (2003) 062 [hep-ph/0302180];
A. Gehrmann-De Ridder, T. Gehrmann and G. Heinrich, *Four-particle phase space integrals in massless QCD*, *Nucl. Phys.* **B 682** (2004) 265 [hep-ph/0311276];
C. Anastasiou, K. Melnikov and F. Petriello, *A new method for real radiation at NNLO*, *Phys. Rev.* **D 69** (2004) 076010 [hep-ph/0311311];
T. Binoth and G. Heinrich, *Numerical evaluation of phase space integrals by sector decomposition*, *Nucl. Phys.* **B 693** (2004) 134 [hep-ph/0402265];
C. Anastasiou, K. Melnikov and F. Petriello, *Real radiation at nnlo: $e^+e^- \to 2$ jets through $O(\alpha_s^2)$*, *Phys. Rev. Lett.* **93** (2004) 032002 [hep-ph/0402280];
A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Infrared structure of $e^+e^- \to 2$ jets at NNLO*, *Nucl. Phys.* **B 691** (2004) 195 [hep-ph/0403057];
W.B. Kilgore, *Subtraction terms for hadronic production processes at next-to-next-to-leading order*, *Phys. Rev.* **D 70** (2004) 031501 [hep-ph/0403128];
A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Infrared structure of $e^+e^- \to$ 3jets at nnlo: the $C_F^2$ contribution*, hep-ph/0407023.